Introduction

Lagrangian singularities first appeared in the work of Arnold and his school around 1980. Arnold recognized their importance in relation with problems from mathematical physics, in particular, variational problems with constraints ([Arn82]). Most prominently, the so-called obstacle problem leads to the open swallowtail, a singular subvariety in a certain space of polynomials in one variable of fixed degree, which comes equipped with a natural symplectic form. Some years later, Givental studied immersions of lagrangian surfaces in four space ([Giv86]), also called isotropic mappings and discovered a generic mapping the image of which is called open Whitney umbrella. More recently, lagrangian subvarieties associated to any *Frobenius* manifold have been studied extensively by Hertling [Her02]. Singular subspaces of symplectic manifolds also arise in algebraic analysis, the characteristic variety of a holonomic $\mathcal{D}$-module is a lagrangian subvariety. These few examples show that Lagrangian singularities occur at rather different places in mathematics, as subspaces of holomorphic symplectic manifolds as well as in the C^∞-setting. There are also classes of lagrangian submanifolds involving real and complex structures, namely the so-called special lagrangians are subspaces of Calabi-Yau manifolds such that the Kähler form as well as the imaginary part of the holomorphic form of maximal degree vanish on them. Singularities of such special lagrangians play an important role in the (conjectural) version of mirror symmetry as developed by Strominger, Yau and Zaslow (see, e.g., [Joy00]).

The central topic of this thesis is the problem how lagrangian singularities behave under deformations. Partial aspects of this question can already be found in the work of Givental ([Giv88]). However, the deformations that are considered in that paper are only perturbations of the symplectic structure which fixes the lagrangian subspace. In order to take into account deformations of the space itself, we are led to use rather sophisticated tools from abstract deformation theory, which have been developed since the sixties (quite independently from classical singularity theory) by Grothendieck, Schlessinger, Illusie, Artin, Deligne and others. In this approach, the main idea is to associate to any object that one wants to deform a functor on a certain category (which is the category of base spaces of the families under consideration) and to study its representability, at least in a somewhat weaker sense (existence of a so-called "hull"). The classical notion of semi-universal deformations (e.g., for functions with isolated critical points) is a special case of this more general principle.

To make this deformation theory program work, the first step is to define the appropriate functor. Hence we need to know what exactly is meant by a Lagrangian deformation. We will give in the sequel an informal definition, postponing the exact formulation to the second chapter (definition 2.4 on page 54). Given any germ $(L, 0) \subset (M, 0)$ of a reduced (complex, say) analytic subspace L inside a (holomorphic) symplectic manifold M with defining ideal $I \subset \mathcal{O}_{M,0}$, the question arises how to detect whether L is lagrangian only in terms of the ideal I. It turns out that a necessary condition is that I is stable under the Poisson bracket, i.e., $\{I, I\} \subset I$. Such ideals are called *involutive*. In addition, the space L must have the right dimension, i.e., half of the dimension of the manifold M. If we want to deform this situation, the first thing to realize is that the ambient manifold should deform trivially and that the deformed space L_S will be embedded in $M \times S$, where S is the parameter space. The condition to impose is that for each $s \in S$, the fibre $L_s \subset M \times \{s\}$ is a lagrangian subvariety. In terms of the defining ideal, this simply means that if $I_S \subset \mathcal{O}_{M\times S,0}$ is the deformed ideal (the ideal defining L_S in $M \times S$), we require that $\{I_S, I_S\} \subset I_S$. Here the bracket is a bracket on the product $M \times S$, this is no longer a symplectic but a Poisson manifold (i.e, the bracket is degenerate). Again, we need a condition on the dimension of the fibres. This is automatic if we require the deformation to be *flat* as usual for singularities. Then all fibres will have the same dimension, namely, half of the dimension of M. Given a deformation of $L_S \subset M \times S \twoheadrightarrow S$, the natural question arises whether it can be trivialized. In the case of flat deformations of (arbitrary) singularities, a trivialization is given by a vector field of the ambient manifold. This is still true for a lagrangian deformation, however, as we are working in the symplectic category, this vector field must be hamiltonian. The description just given already suffices to define our lagrangian deformation functor, namely, it is a functor from an appropriate category of base spaces into the category of sets which associates to a space S the set of isomorphism classes of lagrangian deformations over S modulo isomorphisms coming from Hamiltonian vector fields.

Given a deformation functor, there are in general two things one is interested in. The first one is the existence of a hull (a formally semi-universal deformation). This is a deformation over a space $Spec(R)$ where R is a quotient of a formal power series ring. One of the fundamental results of Schlessinger is that such a hull exists if the space of deformations over $Spec(k[\epsilon]/\epsilon^2)$ (called the *tangent space* of the functor) is a finite-dimensional vector space over k. The second point is to study the structure of the hull R, in particular, to know whether it is smooth or not. This is known as the problem of obstructions, namely, it consists in detecting whether for a deformation over an Artin space $Spec(A)$ and a surjection $B \twoheadrightarrow A$ there is a deformation over $Spec(B)$ inducing the given deformation over A. The most conceptual way to treat these two problems together is to find what is called a "controlling differential graded Lie algebra" $(L, d, [\ ,\ ])$. This roughly means that the space of deformations over a ring A is identified with the subset of $L^1 \otimes \mathbf{m}_A$ consisting of solutions of the following equation, called Maurer-Cartan equation:

$$d\eta + \frac{1}{2}[\eta, \eta] = 0$$

In particular, this implies that the first cohomology $H^1(L)$ is the tangent space of the functor and $H^2(L)$ contains in some sense "all" obstructions.

One case where this theory has been successfully applied is the problem of flat deformations of a singularity $(X, 0)$, that is, flat deformations of the analytic algebra $\mathcal{O}_{X,0}$ (there is of course a corresponding theory in the algebraic category). Here a dg-Lie algebra, constructed from the so-called (analytic) cotangent complex exists. It is a complex of $\mathcal{O}_{X,0}$-modules together with a graded Lie bracket which makes it into a differential graded Lie algebra. Very roughly, it is defined as the complex of graded derivations of a special resolution of $\mathcal{O}_X$ (called the resolvent) where the bracket is the commutator of derivations and the differential is the bracket with the differential of the resolvent (which is a derivation of degree one).

For lagrangian singularities, the situation is more difficult, as one has to take into account both the flatness and the lagrangian condition. We construct in this work for any lagrangian singularity $(L,0) \subset (M,0)$ a complex of $\mathcal{O}_L$-modules (denoted by $\mathcal{C}^{\bullet}_{L,0}$) together with a $\mathbb{C}$-linear differential whose first cohomology is identified with the tangent space of the lagrangian deformation functor. The second cohomology contains information on the obstruction theory of $(L,0)$. However, this complex does not control the deformation problem in the above sense, the main reason is that it is not equipped with a bracket making it into a differential graded Lie algebra. It should be seen as an approximation of an object still to be found.

The complex $\mathcal{C}^{\bullet}_{L,0}$ turns out to be related to the theory of differential modules. This somewhat surprising fact can be explained by the formalism of Lie-algebroids. A Lie algebroid on a space X is a module over $\mathcal{O}_X$ together with a Lie algebra structure, such that elements act as derivations of $\mathcal{O}_X$. For any lagrangian singularity, the conormal module I/I^2 has a natural structure of a Lie algebroid, where the Lie bracket and the action on $\mathcal{O}_{L,0}$ is essentially given by the Poisson bracket. There is a natural construction of a (non-commutative) ring of differential operators from a given Lie algebroid. This construction generalizes the usual ring of differential operators, which comes in the same way from the tangent sheaf of a smooth variety X viewed as a (rather trivial) Lie algebroid. The complex $\mathcal{C}^{\bullet}_{L,0}$ is the analogue of the de Rham complex in $\mathcal{D}$-module theory (therefore we call it lagrangian de Rham complex). The second main result of this work is a version of Kashiwara's constructibility theorem for the lagrangian de Rham complex. In ordinary $\mathcal{D}$-modules theory, this result states that for a *holonomic* $\mathcal{D}_X$-module $\mathcal{M}$, the cohomology of the de Rham complex $DR^{\bullet}(\mathcal{M},\mathcal{O}_X)$ form constructible sheaves of finite-dimensional vector spaces on X. We prove a similar result for the complex $\mathcal{C}_L^\bullet$ under a geometric condition on the lagrangian variety L. This implies in particular by using Schlessinger's theorem the existence of a semi-universal deformation (in the formal sense) for lagrangian singularities satisfying this condition. The relation to the de Rham complex of the space L also yields a sort of $\mu = \tau$ theorem for smoothable lagrangian singularities.

A major problem concerning the deformation spaces of lagrangian singularities was to know how to calculate them effectively. In fact, the description of the tangent space of the lagrangian deformation functor as the first cohomology of $\mathcal{C}_{L,0}^\bullet$ is a priori not sufficient to compute this space. The main difficulty lies in the non-linearity of the differential. Hopefully, a direct calculation might be possible using the differential structure and the theory of standard bases over general non-commutative algebras. This subject is however still in its infancy. Meanwhile, we can offer an algorithm for reduced quasi-homogenous lagrangian surfaces. In that case the computation simplifies to the calculation of the cohomology of a smaller complex, which is supported on the singular locus of L. Then the differential structure is much easier to understand, it reduces essentially to a vector bundle over the complex line together with a meromorphic connection. Classical results from the theory of ordinary differential equations allow us to calculate the space of horizontal sections of this bundle, which gives the cohomology we are interested in. As a byproduct, we obtain a set of rational numbers, the so called spectral numbers which are invariants attached to the lagrangian surface. They are in some sense an analogue to the spectrum of a hypersurface singularity with isolated critical points, which is an important ingredient to define a mixed Hodge structure on the cohomology of the Milnor fibre of the singularity. Quite surprisingly, our lagrangian spectral numbers share a symmetry property with the classical spectrum, at least in all examples we have calculated. For the spectrum of a function with isolated critical points, the symmetry is a deep result using K. Saito's higher residue pairings. For the lagrangian spectrum, the symmetry has not yet been shown. We explain in the text some ideas and speculations which might lead to a rigorous proof.

There is another deformation problem related to lagrangian singularities, namely, deformations of so-called isotropic mappings. Suppose that there is a map from a smooth variety into a symplectic manifold such that the image is a lagrangian subvariety. Then one might ask about the deformations of this map requiring that the image stays lagrangian. This problem turns out to be more difficult to attack than deformations of lagrangian subvarieties, in fact, there is not yet a systematic way to compute these deformation spaces. Nevertheless, we can calculate them for simple examples, like plane curves and isotropic mappings from a plane into four space of rank one. In general, isotropic mappings of corank one are of rather special type, e.g., their deformation functor is smooth, which is not true in general. The calculation of the infinitesimal deformation space of isotropic mappings from a plane into four space shows an astonishing relation between the dimension of this space and other (more classical) invariants attached to the map. We conjecture that this relation holds true in general.

We will give in the following paragraphs a short overview on the content of this thesis. The first chapter describes in some detail the geometry of different classes of lagrangian singularities. Apart from the examples mentioned above we discuss generating families, integrable systems, the $\mu/2$-stratum, spectral covers of Frobenius manifolds and singularities of special lagrangian varieties. We present for each of these classes one example as concrete as possible (mainly the case of a surface in four-space) by calculating a set of defining equations $f_1, \ldots, f_k$, the commutator $\{f_i, f_j\}$ of these equations, the structure of the singular locus etc. Despite the fact that these examples are well-known, this type of calculations (using computer algebra) is difficult to find in the literature.

The second chapter introduces the problem of deformations in the lagrangian context by first studying two very simple examples, which are in some sense opposite to each other: smooth real lagrangian submanifolds of C^∞-manifolds and germs of plane curves. Here it is elementary to calculate infinitesimal deformation spaces, these are classical results. Then we introduce a quite general deformation functor, associated to any mapping $i : X \to M$ from an analytic space to a symplectic manifold such that $i^*\omega$ vanishes. For a lagrangian subvariety, one can take i to be the inclusion to obtain the functor mentioned above. On the other hand, if X is smooth and i arbitrary then we get the functor of deformations of an isotropic mapping. These two cases are treated in detail in the following two chapters. The third one starts by introducing Lie algebroids and modules over them. We define the de Rham complex of a module over a Lie algebroid. Then we prove that the conormal module of a lagrangian subvariety $L \subset M$ has the structure of a Lie algebroid. We study simple properties of the lagrangian de Rham complex $\mathcal{C}_L^\bullet$, in particular, we compare it to several complexes of differential forms on the variety L. We introduce the whole theory directly in a relative setting, that is, we define Lie algebroids over morphisms of analytic spaces. This situation arises naturally by considering a *family* $\mathcal{L} \hookrightarrow M \times S \twoheadrightarrow S$ of lagrangian varieties over a base S. The next step is to prove that the first cohomology of the lagrangian de Rham complex is isomorphic to the tangent space of the lagrangian deformation functor (again, this is done in a relative setting, considering infinitesimal deformations of the family). We state and show a variant of a T^1-lifting theorem for lagrangian singularities which gives the smoothness of the deformation functors in some cases. Finally, we discuss a slightly modified deformation problem concerning integrable systems. Here we have a more complete result, we can construct from the lagrangian de Rham complex a differential graded Lie algebra controlling deformations of integrable systems.

The second part of the third chapter contains the proof of the constructibility theorem. It follows the proof of Kashiwara's theorem for $\mathcal{D}$-modules, namely, we first show that the cohomology sheaves of the complex $\mathcal{C}_L^\bullet$ are locally constant on strata consisting of points of L with constant embedding dimension. The second step is to show that at each point $p \in L$, the stalk of a cohomology sheaf is a finite dimensional vector space. This part uses an idea from functional analysis (the Kiehl-Verdier theorem) which was already the key ingredient for similar finiteness results in different situations (e.g., [BG80]). The main geometric argument for both parts of this proof is the following: Let $p \in L$ a point and consider the germ (L, p) of L at p, which is of dimension n. Its embedding dimension might vary in between n and $2n$. If it is strictly smaller than $2n$, then the variety is locally around p a product $L = L' \times C$, where C is a smooth curve, and L' is a lagrangian subspace in a symplectic manifold of dimension $2n - 2$. This is already found in [Giv88]. Now the main point is that such a lagrangian product is rather rigid, it can only be deformed as a product by deforming the factor L'. We call this principle propagation of deformations. Globally, it implies that if the points of L of maximal embedding dimension are isolated (this is essentially the assumption for our constructibility theorem), then the cohomology of $\mathcal{C}_L^\bullet$ over a small neighborhood of such a point will not change if we restrict to a smaller neighborhood. By the theorem of Kiehl-Verdier its stalk at this point must be finite-dimensional. Lagrangian singularities having isolated points with maximal embedding dimension therefore have a (formally) semi-universal deformation. Hence singularities satisfying this condition are the lagrangian analogue to isolated singularities. We finish the second chapter by explaining our method of computing the cohomology of $\mathcal{C}_L^\bullet$ for a quasi-homogeneous surface. We introduce the spectral numbers and make some conjectures concerning their symmetry.

The last chapter treats isotropic mappings. After introducing basic properties of their deformation spaces, we calculate the tangent space of its deformation functor for monomial curves and for maps having as its image a lagrangian singularity which can be decomposed into a lagrangian singularity of smaller dimension and a smooth space. Here there is no such rigidity theorem as for deformations of subvarieties. Therefore in general versal deformations of isotropic maps will exist only if the critical values are isolated. We discuss in detail one particular isotropic map, the normalization of the open Whitney umbrella. It was already known that this map is rigid. Moreover, there is the following theorem, stated (and proved in particular cases) by Givental ([Giv86]) and shown in general by Ishikawa ([Ish92]): Consider the space of germs of isotropic maps form $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, equipped with the Whitney C^∞-topology. Then this space contains a dense open subset of maps which are equivalent (modulo diffeomorphisms of $\mathbb{R}^n$ and symplectomorphisms of $\mathbb{R}^{2n}$) to a generalized open Whitney umbrella (which is the usual one for $n = 2$). This result is briefly reviewed. We finish this chapter by calculating the dimension of the infinitesimal lagrangian deformation space as well as the δ-invariant, the usual infinitesimal deformation space and the dimension of the module of relative differential forms for corank one maps from $\mathbb{R}^2$ into $\mathbb{R}^4$. We conjecture a linear relation between some of these numbers.

We have included two appendices in this thesis. The first (rather large) one reviews the concepts of abstract deformation theory that are used in the text. As there is not yet a standard reference for this theory, it seems appropriate to collect the results we need. We discuss first deformation functors and categories fibred in groupoids as well as differential graded Lie algebras. We define the notion of a controlling dg-Lie algebra. Finally, the so called T^1-lifting theorem is stated and proved. This is a tool to deduce smoothness of a functor from a certain lifting property of its relative tangent spaces.

In the second part of this appendix we describe basic examples of controlling dg-Lie algebras. These include deformations of complex structures, associative algebras and flat deformations of analytic algebras. The latter involves the cotangent complex, which we review in some detail.

The second appendix is a very brief introduction to the theory of differential modules. The aim is to define notions and principles which are used (mainly while developing the analogous versions for general Lie algebroids) in the text. We define the ring $\mathcal{D}_X$, modules over it, good filtrations and coherent $\mathcal{D}$-modules, the characteristic variety and holonomic $\mathcal{D}$-modules. We prove Kashiwara's constructibility theorem in complete analogy with our proof for the lagrangian de Rham complex.

Let us finish this introduction by listing some problems and questions related to lagrangian singularities which are still open or only partially answered. We already mentioned the problem of finding a controlling dg-Lie algebra for the functor of deformations of a lagrangian subvariety. It should incorporate the cotangent complex in some way because our lagrangian deformations are flat by definition. On the other hand, even the question whether for an ideal which is involutive up to order n there is a lift to an ideal involutive up to order $n+1$ cannot be answered directly from the complex $\mathcal{C}_L^\bullet$. There should be a graded bracket on this complex derived from the Poisson bracket which gives the obstruction map. The difficulty comes from the fact that the Poisson bracket (defined on $\mathcal{O}_M$) does not descend to $\mathcal{O}_L$. See theorem 3.20 on page 74 for more details.

The symmetry of the spectrum for a lagrangian surface singularity is probably related to the existence of a naturally given bilinear form on a meromorphic bundle, which comes from the quotient of the lagrangian de Rham complex by the de Rham complex of ordinary differential forms on the variety. This quotient is supported on the singular locus, and we expect that it can be identified with a bundle the fibre of which at a point is isomorphic to the cohomology of the Milnor fibre of the transversal singularity at this point. However, this bundle must be defined canonically, without choosing local coordinates. This is still to be done.

Another open question concerns the structure of the category of modules over the Lie algebroid I/I^2 (the conormal module). At least in the case when this module is locally free (i.e., for complete intersections), things are easier to handle and it is likely that the ring of generalized differential operators constructed from I/I^2 is of finite homological dimension. In principle, the corresponding proof for ordinary $\mathcal{D}$-modules can be adapted to this more general situation. However, the crucial ingredient is a dimension estimate using the Bernstein inequality for the dimension of the characteristic variety. The characteristic variety of a $\mathcal{D}_X$-module is a subspace of the cotangent bundle T^*X. In our case, there is an analogue of the cotangent bundle, namely, a linear space $\mathcal{S}$ over the variety L and the algebra $\mathcal{O}_\mathcal{S}$ is equipped with a Poisson bracket. But S is itself singular (because L is singular), so it is not a symplectic manifold and it might be difficult to estimate the dimension of the characteristic variety.

Returning to deformation theory, it should be noticed that although we define all objects globally, i.e., for a lagrangian subspace of a symplectic manifold, our results are local in nature. We study essentially deformations of germs (or small representatives of them). The global deformation theory is probably also controlled by the lagrangian de Rham complex, e.g., the infinitesimal deformations are given by the first *hypercohomology* of this complex. This is however not so easy to see, much like in the case of flat deformations, where rather heavy machinery (simplicial resolutions of complex spaces) is needed to study global deformations.

Let $\mathcal{L} \to S$ be a lagrangian deformation over a base S where $\mathcal{O}_{S,0}$ is an *analytic* algebra. Suppose that it is infinitesimal versal, i.e., the tangent space of S at zero is isomorphic to the tangent space of the deformation functor. In this situation one would like to know whether the family is versal in the strong sense, i.e., whether every deformation is equivalent by an *analytic change of coordinate* to a deformation induced from $\mathcal{L} \to S$. For flat deformations, a semi-universal deformation in this sense exists if the singularities are isolated, this is Grauert's theorem. It uses approximation techniques in order to obtain convergent solutions. For lagrangian singularities, there is not yet such a complete picture. We can give a stability theorem for a family as above. This result is due to M. Garay ([Gar02]) in the case of complete intersections. We introduce a *Kodaira-Spencer* map to apply it in general. However, the convergency of versal deformations in general is unknown. A simple use of Grauert's approximation theorem will not be sufficient, because we need that the analytic coordinate change stays symplectic.

A last remark concerning the comparison of the different categories we are working in seems in order. In application (involving the classes of examples that we treat in the first chapter), one encounters both symplectic manifolds of class C^∞ and holomorphic symplectic manifolds. In the real case one may consider C^∞- or analytic lagrangian submanifolds. In order to give a unified treatment, we adopt the following terminology: Symplectic manifolds over $\mathbb{K}$ which denotes either $\mathbb{R}$ or $\mathbb{C}$ are C^∞- or holomorphic symplectic manifolds, respectively. We work only with analytic lagrangian submanifolds in both cases. For some of our results we need to restrict to the complex case, in particular, for the constructibility theorem. One can always consider the complexification of a real analytic lagrangian subspace. However, this may introduce additional conditions of the complex part on the variety not visible over $\mathbb{R}$.