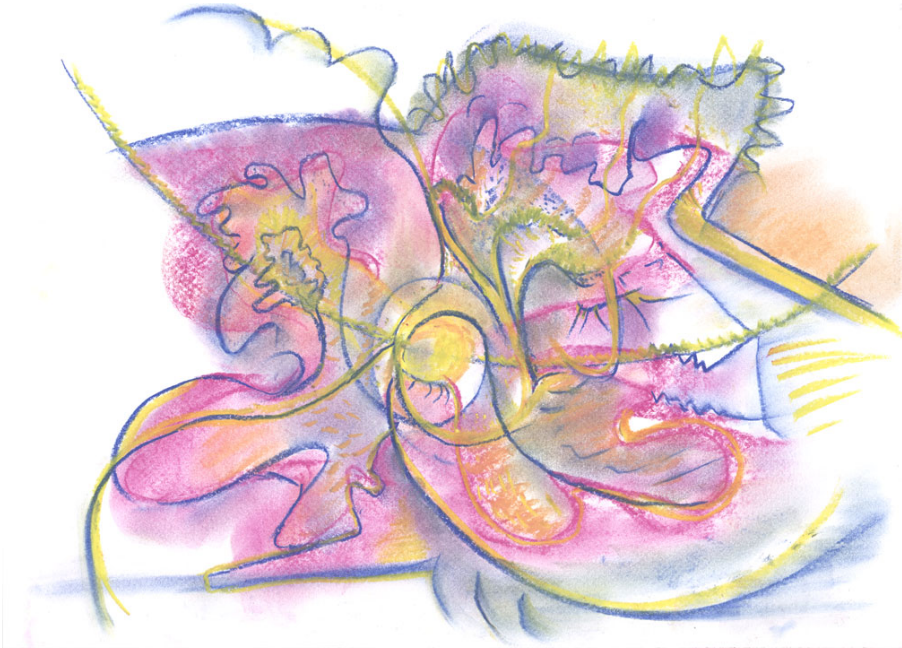
Christian Sevenheck

Lagrangian Singularities



Cuvillier Verlag Göttingen

Lagrange-Singularitäten

Dissertation zur Erlangung des Grades
„Doktor der Naturwissenschaften“
am Fachbereich Mathematik
der Johannes-Gutenberg-Universität in Mainz
vorgelegt von
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Tag der mündlichen Prüfung: 27. Januar 2003

Titelbild: Elsa Martin

Thèse presentée pour obtenir le titre de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Spécialité:

Mathématiques

par

Christian SEVENHECK

Titre: Singularités lagrangiennes

soutenue le 27 janvier 2003 devant le jury composée de:

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Preface

In this work, based mainly on my Ph.D. thesis, lagrangian singularities are studied. This topic lies on the border of different branches of mathematics, like singularity theory and algebraic geometry, symplectic geometry, mathematical physics, algebraic analysis etc. The main goal is to develop a deformation theory for lagrangian singularities and to investigate its relationship to $\mathcal{D}$-module theory. Algorithms for computations of deformation spaces are derived and applied to concrete examples.

It is a great pleasure for me to acknowledge the help I received from different people during the work on my thesis. In the first place, I would like to express my deep gratitude to my advisor Duco van Straten for his constant support which began a long time before I started this work. His way to do and explain mathematics very much impressed and influenced me all over the years. I am particular grateful to him for bringing the subject of this thesis to my attention and for stimulating a good part of the ideas contained herein. Secondly, I like to thank Claude Sabbah who kindly accepted to undertake the adventure of co-directing my thesis and whose interest in my work as well as his explanations on $\mathcal{D}$-modules were of great help during my stay in Paris.

It is tempting but hopeless to try to list all the people who contributed in some way to this thesis. To name only a few, I thank Thorsten Warmt for sharing my enthusiasm for mathematics over the last eight years. I also thank Christian van Enckevort, Konrad Möhring and Oliver Labs for discussions on various mathematical subjects. Many thanks go to Claus Hertling for explaining me at different occasions his work on Frobenius manifolds and the relationship to lagrangian singularities. It is a pleasure to thank Mauricio Garay for his interest in lagrangian singularities and for many fruitful discussion on the subject, some of which are at the origin of results contained in this thesis.

I would like to thank Paul Seidel and Gert-Martin Greuel for having been the referees of this thesis and for their useful remarks. I thank Claude Roger for his willingness to participate in the jury of the defense.

This work was done while I was enrolled as a Ph.D. student both at the Fachbereich Mathematik of the Johannes-Gutenberg-Universität Mainz and at the Centre de Mathématiques, Ecole Polytechnique. I would like to thank these institutions for the excellent working conditions they provided. In particular, I thank Renate Emerenziani who helped me a lot handling all type of administrative difficulties.

Schließlich möchte ich meiner Freundin Elsa Martin dafür danken, dass sie wärend der langen Jahre, in denen ich mich mit dieser Arbeit beschäftigte, immer für mich da war und mich dann, wenn ich glaubte, nicht mehr weiterzukommen, ermutigt und wiederaufgerichtet hat. Meiner Mutter danke ich dafür, dass sie mich seit langem in vieler Hinsicht unterstützt und mir mein Studium überhaupt erst ermöglicht hat.

Mainz, 2nd June 2003 Christian Sevenheck

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Introduction

Lagrangian singularities first appeared in the work of Arnold and his school around 1980. Arnold recognized their importance in relation with problems from mathematical physics, in particular, variational problems with constraints ([Arn82]). Most prominently, the so-called obstacle problem leads to the open swallowtail, a singular subvariety in a certain space of polynomials in one variable of fixed degree, which comes equipped with a natural symplectic form. Some years later, Givental studied immersions of lagrangian surfaces in four space ([Giv86]), also called isotropic mappings and discovered a generic mapping the image of which is called open Whitney umbrella. More recently, lagrangian subvarieties associated to any *Frobenius* manifold have been studied extensively by Hertling [Her02]. Singular subspaces of symplectic manifolds also arise in algebraic analysis, the characteristic variety of a holonomic $\mathcal{D}$-module is a lagrangian subvariety. These few examples show that Lagrangian singularities occur at rather different places in mathematics, as subspaces of holomorphic symplectic manifolds as well as in the C^∞-setting. There are also classes of lagrangian submanifolds involving real and complex structures, namely the so-called special lagrangians are subspaces of Calabi-Yau manifolds such that the Kähler form as well as the imaginary part of the holomorphic form of maximal degree vanish on them. Singularities of such special lagrangians play an important role in the (conjectural) version of mirror symmetry as developed by Strominger, Yau and Zaslow (see, e.g., [Joy00]).

The central topic of this thesis is the problem how lagrangian singularities behave under deformations. Partial aspects of this question can already be found in the work of Givental ([Giv88]). However, the deformations that are considered in that paper are only perturbations of the symplectic structure which fixes the lagrangian subspace. In order to take into account deformations of the space itself, we are led to use rather sophisticated tools from abstract deformation theory, which have been developed since the sixties (quite independently from classical singularity theory) by Grothendieck, Schlessinger, Illusie, Artin, Deligne and others. In this approach, the main idea is to associate to any object that one wants to deform a functor on a certain category (which is the category of base spaces of the families under consideration) and to study its representability, at least in a somewhat weaker sense (existence of a so-called “hull”). The classical notion of semi-universal deformations (e.g., for functions with isolated critical points) is a special case of this more general principle.

To make this deformation theory program work, the first step is to define the appropriate functor. Hence we need to know what exactly is meant by a Lagrangian deformation. We will give in the sequel an informal definition, postponing the exact formulation to the second chapter (definition 2.4 on page 54). Given any germ $(L, 0) \subset (M, 0)$ of a reduced (complex, say) analytic subspace L inside a (holomorphic) symplectic manifold M with defining ideal $I \subset \mathcal{O}_{M,0}$, the question arises how to detect whether L is lagrangian only in terms of the ideal I. It turns out that a necessary condition is that I is stable under the Poisson bracket, i.e., $\{I, I\} \subset I$. Such ideals are called *involutive.* In addition, the space L must have the right dimension, i.e., half of the dimension of the manifold M. If we want to deform this situation, the first thing to realize is that the ambient manifold should deform trivially and that the deformed space L_S will be embedded in $M \times S$, where S is the parameter space. The condition to impose is that for each $s \in S$, the fibre $L_s \subset M \times \{s\}$ is a lagrangian subvariety. In terms of the defining ideal, this simply means that if $I_S \subset \mathcal{O}_{M\times S,0}$ is the deformed ideal (the ideal defining L_S in $M \times S$), we require that $\{I_S, I_S\} \subset I_S$. Here the bracket is a bracket on the product $M \times S$, this is no longer a symplectic but a Poisson manifold (i.e, the bracket is degenerate). Again, we need a condition on the dimension of the fibres. This is automatic if we require the deformation to be *flat* as usual for singularities. Then all fibres will have the same dimension, namely, half of the dimension of M. Given a deformation of $L_S \subset M \times S \twoheadrightarrow S$, the natural question arises whether it can be trivialized. In the case of flat deformations of (arbitrary) singularities, a trivialization is given by a vector field of the ambient manifold. This is still true for a lagrangian deformation, however, as we are working in the symplectic category, this vector field must be hamiltonian. The description just given already suffices to define our lagrangian deformation functor, namely, it is a functor from an appropriate category of base spaces into the category of sets which associates to a space S the set of isomorphism classes of lagrangian deformations over S modulo isomorphisms coming from Hamiltonian vector fields.

Given a deformation functor, there are in general two things one is interested in. The first one is the existence of a hull (a formally semi-universal deformation). This is a deformation over a space $Spec(R)$ where R is a quotient of a formal power series ring. One of the fundamental results of Schlessinger is that such a hull exists if the space of deformations over $Spec(k[\epsilon]/\epsilon^2)$ (called the *tangent space* of the functor) is a finite-dimensional vector space over k. The second point is to study the structure of the hull R, in particular, to know whether it is smooth or not. This is known as the problem of obstructions, namely, it consists in detecting whether for a deformation over an Artin space $Spec(A)$ and a surjection $B \twoheadrightarrow A$ there is a deformation over $Spec(B)$ inducing the given deformation over A. The most conceptual way to treat these two problems together is to find what is called a "controlling differential graded Lie algebra" $(L, d, [\ ,\ ])$. This roughly means that the space of deformations over a ring A is identified with the subset of $L^1 \otimes \mathbf{m}_A$ consisting of solutions of the following equation, called Maurer-Cartan equation:

$$d\eta + \frac{1}{2}[\eta, \eta] = 0$$

In particular, this implies that the first cohomology $H^1(L)$ is the tangent space of the functor and $H^2(L)$ contains in some sense "all" obstructions.

One case where this theory has been successfully applied is the problem of flat deformations of a singularity $(X, 0)$, that is, flat deformations of the analytic algebra $\mathcal{O}_{X,0}$ (there is of course a corresponding theory in the algebraic category). Here a dg-Lie algebra, constructed from the so-called (analytic) cotangent complex exists. It is a complex of $\mathcal{O}_{X,0}$-modules together with a graded Lie bracket which makes it into a differential graded Lie algebra. Very roughly, it is defined as the complex of graded derivations of a special resolution of $\mathcal{O}_X$ (called the resolvent) where the bracket is the commutator of derivations and the differential is the bracket with the differential of the resolvent (which is a derivation of degree one).

For lagrangian singularities, the situation is more difficult, as one has to take into account both the flatness and the lagrangian condition. We construct in this work for any lagrangian singularity $(L,0) \subset (M,0)$ a complex of $\mathcal{O}_L$-modules (denoted by $\mathcal{C}^\bullet_{L,0}$) together with a $\mathbb{C}$-linear differential whose first cohomology is identified with the tangent space of the lagrangian deformation functor. The second cohomology contains information on the obstruction theory of $(L,0)$. However, this complex does not control the deformation problem in the above sense, the main reason is that it is not equipped with a bracket making it into a differential graded Lie algebra. It should be seen as an approximation of an object still to be found.

The complex $\mathcal{C}^\bullet_{L,0}$ turns out to be related to the theory of differential modules. This somewhat surprising fact can be explained by the formalism of Lie-algebroids. A Lie algebroid on a space X is a module over $\mathcal{O}_X$ together with a Lie algebra structure, such that elements act as derivations of $\mathcal{O}_X$. For any lagrangian singularity, the conormal module I/I^2 has a natural structure of a Lie algebroid, where the Lie bracket and the action on $\mathcal{O}_{L,0}$ is essentially given by the Poisson bracket. There is a natural construction of a (non-commutative) ring of differential operators from a given Lie algebroid. This construction generalizes the usual ring of differential operators, which comes in the same way from the tangent sheaf of a smooth variety X viewed as a (rather trivial) Lie algebroid. The complex $\mathcal{C}^\bullet_{L,0}$ is the analogue of the de Rham complex in $\mathcal{D}$-module theory (therefore we call it lagrangian de Rham complex). The second main result of this work is a version of Kashiwara's constructibility theorem for the lagrangian de Rham complex. In ordinary $\mathcal{D}$-modules theory, this result states that for a *holonomic* $\mathcal{D}_X$-module $\mathcal{M}$, the cohomology of the de Rham complex $DR^\bullet(\mathcal{M},\mathcal{O}_X)$ form constructible sheaves of finite-dimensional vector spaces on X. We prove a similar result for the complex $\mathcal{C}_L^\bullet$ under a geometric condition on the lagrangian variety L. This implies in particular by using Schlessinger's theorem the existence of a semi-universal deformation (in the formal sense) for lagrangian singularities satisfying this condition. The relation to the de Rham complex of the space L also yields a sort of $\mu = \tau$ theorem for smoothable lagrangian singularities.

A major problem concerning the deformation spaces of lagrangian singularities was to know how to calculate them effectively. In fact, the description of the tangent space of the lagrangian deformation functor as the first cohomology of $\mathcal{C}_{L,0}^\bullet$ is a priori not sufficient to compute this space. The main difficulty lies in the non-linearity of the differential. Hopefully, a direct calculation might be possible using the differential structure and the theory of standard bases over general non-commutative algebras. This subject is however still in its infancy. Meanwhile, we can offer an algorithm for reduced quasi-homogenous lagrangian surfaces. In that case the computation simplifies to the calculation of the cohomology of a smaller complex, which is supported on the singular locus of L. Then the differential structure is much easier to understand, it reduces essentially to a vector bundle over the complex line together with a meromorphic connection. Classical results from the theory of ordinary differential equations allow us to calculate the space of horizontal sections of this bundle, which gives the cohomology we are interested in. As a byproduct, we obtain a set of rational numbers, the so called spectral numbers which are invariants attached to the lagrangian surface. They are in some sense an analogue to the spectrum of a hypersurface singularity with isolated critical points, which is an important ingredient to define a mixed Hodge structure on the cohomology of the Milnor fibre of the singularity. Quite surprisingly, our lagrangian spectral numbers share a symmetry property with the classical spectrum, at least in all examples we have calculated. For the spectrum of a function with isolated critical points, the symmetry is a deep result using K. Saito's higher residue pairings. For the lagrangian spectrum, the symmetry has not yet been shown. We explain in the text some ideas and speculations which might lead to a rigorous proof.

There is another deformation problem related to lagrangian singularities, namely, deformations of so-called isotropic mappings. Suppose that there is a map from a smooth variety into a symplectic manifold such that the image is a lagrangian subvariety. Then one might ask about the deformations of this map requiring that the image stays lagrangian. This problem turns out to be more difficult to attack than deformations of lagrangian subvarieties, in fact, there is not yet a systematic way to compute these deformation spaces. Nevertheless, we can calculate them for simple examples, like plane curves and isotropic mappings from a plane into four space of rank one. In general, isotropic mappings of corank one are of rather special type, e.g., their deformation functor is smooth, which is not true in general. The calculation of the infinitesimal deformation space of isotropic mappings from a plane into four space shows an astonishing relation between the dimension of this space and other (more classical) invariants attached to the map. We conjecture that this relation holds true in general.

We will give in the following paragraphs a short overview on the content of this thesis. The first chapter describes in some detail the geometry of different classes of lagrangian singularities. Apart from the examples mentioned above we discuss generating families, integrable systems, the $\mu/2$-stratum, spectral covers of Frobenius manifolds and singularities of special lagrangian varieties. We present for each of these classes one example as concrete as possible (mainly the case of a surface in four-space) by calculating a set of defining equations $f_1, \ldots, f_k$, the commutator $\{f_i, f_j\}$ of these equations, the structure of the singular locus etc. Despite the fact that these examples are well-known, this type of calculations (using computer algebra) is difficult to find in the literature.

The second chapter introduces the problem of deformations in the lagrangian context by first studying two very simple examples, which are in some sense opposite to each other: smooth real lagrangian submanifolds of C^∞-manifolds and germs of plane curves. Here it is elementary to calculate infinitesimal deformation spaces, these are classical results. Then we introduce a quite general deformation functor, associated to any mapping $i : X \to M$ from an analytic space to a symplectic manifold such that $i^*\omega$ vanishes. For a lagrangian subvariety, one can take i to be the inclusion to obtain the functor mentioned above. On the other hand, if X is smooth and i arbitrary then we get the functor of deformations of an isotropic mapping. These two cases are treated in detail in the following two chapters. The third one starts by introducing Lie algebroids and modules over them. We define the de Rham complex of a module over a Lie algebroid. Then we prove that the conormal module of a lagrangian subvariety $L \subset M$ has the structure of a Lie algebroid. We study simple properties of the lagrangian de Rham complex $\mathcal{C}_L^\bullet$, in particular, we compare it to several complexes of differential forms on the variety L. We introduce the whole theory directly in a relative setting, that is, we define Lie algebroids over morphisms of analytic spaces. This situation arises naturally by considering a *family* $\mathcal{L} \hookrightarrow M \times S \twoheadrightarrow S$ of lagrangian varieties over a base S. The next step is to prove that the first cohomology of the lagrangian de Rham complex is isomorphic to the tangent space of the lagrangian deformation functor (again, this is done in a relative setting, considering infinitesimal deformations of the family). We state and show a variant of a T^1-lifting theorem for lagrangian singularities which gives the smoothness of the deformation functors in some cases. Finally, we discuss a slightly modified deformation problem concerning integrable systems. Here we have a more complete result, we can construct from the lagrangian de Rham complex a differential graded Lie algebra controlling deformations of integrable systems.

The second part of the third chapter contains the proof of the constructibility theorem. It follows the proof of Kashiwara's theorem for $\mathcal{D}$-modules, namely, we first show that the cohomology sheaves of the complex $\mathcal{C}_L^\bullet$ are locally constant on strata consisting of points of L with constant embedding dimension. The second step is to show that at each point $p \in L$, the stalk of a cohomology sheaf is a finite dimensional vector space. This part uses an idea from functional analysis (the Kiehl-Verdier theorem) which was already the key ingredient for similar finiteness results in different situations (e.g., [BG80]). The main geometric argument for both parts of this proof is the following: Let $p \in L$ a point and consider the germ (L, p) of L at p, which is of dimension n. Its embedding dimension might vary in between n and $2n$. If it is strictly smaller than $2n$, then the variety is locally around p a product $L = L' \times C$, where C is a smooth curve, and L' is a lagrangian subspace in a symplectic manifold of dimension $2n-2$. This is already found in [Giv88]. Now the main point is that such a lagrangian product is rather rigid, it can only be deformed as a product by deforming the factor L'. We call this principle propagation of deformations. Globally, it implies that if the points of L of maximal embedding dimension are isolated (this is essentially the assumption for our constructibility theorem), then the cohomology of $\mathcal{C}_L^\bullet$ over a small neighborhood of such a point will not change if we restrict to a smaller neighborhood. By the theorem of Kiehl-Verdier its stalk at this point must be finite-dimensional. Lagrangian singularities having isolated points with maximal embedding dimension therefore have a (formally) semi-universal deformation. Hence singularities satisfying this condition are the lagrangian analogue to isolated singularities. We finish the second chapter by explaining our method of computing the cohomology of $\mathcal{C}_L^\bullet$ for a quasi-homogeneous surface. We introduce the spectral numbers and make some conjectures concerning their symmetry.

The last chapter treats isotropic mappings. After introducing basic properties of their deformation spaces, we calculate the tangent space of its deformation functor for monomial curves and for maps having as its image a lagrangian singularity which can be decomposed into a lagrangian singularity of smaller dimension and a smooth space. Here there is no such rigidity theorem as for deformations of subvarieties. Therefore in general versal deformations of isotropic maps will exist only if the critical values are isolated. We discuss in detail one particular isotropic map, the normalization of the open Whitney umbrella. It was already known that this map is rigid. Moreover, there is the following theorem, stated (and proved in particular cases) by Givental ([Giv86]) and shown in general by Ishikawa ([Ish92]): Consider the space of germs of isotropic maps form $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, equipped with the Whitney C^∞-topology. Then this space contains a dense open subset of maps which are equivalent (modulo diffeomorphisms of $\mathbb{R}^n$ and symplectomorphisms of $\mathbb{R}^{2n}$) to a generalized open Whitney umbrella (which is the usual one for $n = 2$). This result is briefly reviewed. We finish this chapter by calculating the dimension of the infinitesimal lagrangian deformation space as well as the δ-invariant, the usual infinitesimal deformation space and the dimension of the module of relative differential forms for corank one maps from $\mathbb{R}^2$ into $\mathbb{R}^4$. We conjecture a linear relation between some of these numbers.

We have included two appendices in this thesis. The first (rather large) one reviews the concepts of abstract deformation theory that are used in the text. As there is not yet a standard reference for this theory, it seems appropriate to collect the results we need. We discuss first deformation functors and categories fibred in groupoids as well as differential graded Lie algebras. We define the notion of a controlling dg-Lie algebra. Finally, the so called T^1-lifting theorem is stated and proved. This is a tool to deduce smoothness of a functor from a certain lifting property of its relative tangent spaces.

In the second part of this appendix we describe basic examples of controlling dg-Lie algebras. These include deformations of complex structures, associative algebras and flat deformations of analytic algebras. The latter involves the cotangent complex, which we review in some detail.

The second appendix is a very brief introduction to the theory of differential modules. The aim is to define notions and principles which are used (mainly while developing the analogous versions for general Lie algebroids) in the text. We define the ring $\mathcal{D}_X$, modules over it, good filtrations and coherent $\mathcal{D}$-modules, the characteristic variety and holonomic $\mathcal{D}$-modules. We prove Kashiwara's constructibility theorem in complete analogy with our proof for the lagrangian de Rham complex.

Let us finish this introduction by listing some problems and questions related to lagrangian singularities which are still open or only partially answered. We already mentioned the problem of finding a controlling dg-Lie algebra for the functor of deformations of a lagrangian subvariety. It should incorporate the cotangent complex in some way because our lagrangian deformations are flat by definition. On the other hand, even the question whether for an ideal which is involutive up to order n there is a lift to an ideal involutive up to order $n+1$ cannot be answered directly from the complex $\mathcal{C}^\bullet_L$. There should be a graded bracket on this complex derived from the Poisson bracket which gives the obstruction map. The difficulty comes from the fact that the Poisson bracket (defined on $\mathcal{O}_M$) does not descend to $\mathcal{O}_L$. See theorem 3.20 on page 74 for more details.

The symmetry of the spectrum for a lagrangian surface singularity is probably related to the existence of a naturally given bilinear form on a meromorphic bundle, which comes from the quotient of the lagrangian de Rham complex by the de Rham complex of ordinary differential forms on the variety. This quotient is supported on the singular locus, and we expect that it can be identified with a bundle the fibre of which at a point is isomorphic to the cohomology of the Milnor fibre of the transversal singularity at this point. However, this bundle must be defined canonically, without choosing local coordinates. This is still to be done.

Another open question concerns the structure of the category of modules over the Lie algebroid I/I^2 (the conormal module). At least in the case when this module is locally free (i.e., for complete intersections), things are easier to handle and it is likely that the ring of generalized differential operators constructed from I/I^2 is of finite homological dimension. In principle, the corresponding proof for ordinary $\mathcal{D}$-modules can be adapted to this more general situation. However, the crucial ingredient is a dimension estimate using the Bernstein inequality for the dimension of the characteristic variety. The characteristic variety of a $\mathcal{D}_X$-module is a subspace of the cotangent bundle T^*X. In our case, there is an analogue of the cotangent bundle, namely, a linear space $\mathcal{S}$ over the variety L and the algebra $\mathcal{O}_{\mathcal{S}}$ is equipped with a Poisson bracket. But S is itself singular (because L is singular), so it is not a symplectic manifold and it might be difficult to estimate the dimension of the characteristic variety.

Returning to deformation theory, it should be noticed that although we define all objects globally, i.e., for a lagrangian subspace of a symplectic manifold, our results are local in nature. We study essentially deformations of germs (or small representatives of them). The global deformation theory is probably also controlled by the lagrangian de Rham complex, e.g., the infinitesimal deformations are given by the first *hypercohomology* of this complex. This is however not so easy to see, much like in the case of flat deformations, where rather heavy machinery (simplicial resolutions of complex spaces) is needed to study global deformations.

Let $\mathcal{L} \to S$ be a lagrangian deformation over a base S where $\mathcal{O}_{S,0}$ is an *analytic* algebra. Suppose that it is infinitesimal versal, i.e., the tangent space of S at zero is isomorphic to the tangent space of the deformation functor. In this situation one would like to know whether the family is versal in the strong sense, i.e., whether every deformation is equivalent by an *analytic change of coordinate* to a deformation induced from $\mathcal{L} \to S$. For flat deformations, a semi-universal deformation in this sense exists if the singularities are isolated, this is Grauert's theorem. It uses approximation techniques in order to obtain convergent solutions. For lagrangian singularities, there is not yet such a complete picture. We can give a stability theorem for a family as above. This result is due to M. Garay ([Gar02]) in the case of complete intersections. We introduce a *Kodaira-Spencer* map to apply it in general. However, the convergency of versal deformations in general is unknown. A simple use of Grauert's approximation theorem will not be sufficient, because we need that the analytic coordinate change stays symplectic.

A last remark concerning the comparison of the different categories we are working in seems in order. In application (involving the classes of examples that we treat in the first chapter), one encounters both symplectic manifolds of class C^∞ and holomorphic symplectic manifolds. In the real case one may consider C^∞- or analytic lagrangian submanifolds. In order to give a unified treatment, we adopt the following terminology: Symplectic manifolds over $\mathbb{K}$ which denotes either $\mathbb{R}$ or $\mathbb{C}$ are C^∞- or holomorphic symplectic manifolds, respectively. We work only with analytic lagrangian submanifolds in both cases. For some of our results we need to restrict to the complex case, in particular, for the constructibility theorem. One can always consider the complexification of a real analytic lagrangian subspace. However, this may introduce additional conditions of the complex part on the variety not visible over $\mathbb{R}$.

Chapter 1

Examples of lagrangian singularities

1.1 Involutive ideals and generating families

Throughout this thesis, we will consider symplectic manifolds over the real or complex numbers (we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$). In the complex case, we consider only *holomorphic symplectic* manifolds, i.e., complex manifolds M with a non-degenerate closed two-form ω which lies in $H^0(M, \Omega^2_M)$. Hamiltonian vector fields and Poisson brackets are defined as usual, i.e., for a function $f \in \mathcal{O}_M$ the field $H_f \in \Theta_M$ is defined by $\omega(H_f, Y) = df(Y)$ for all $Y \in \Theta_M$. For any two functions $f, g \in \mathcal{O}_M$ we set $\{f, g\} := \omega(H_f, H_g) = H_g(f)$. We call a reduced analytic subspace L (i.e., a real analytic space resp. a complex space) a **lagrangian subvariety** iff $\omega_{|L_{reg}}$ vanishes, where L_{reg} is the non-singular part of the variety L. A germ $(L, p) \subset (M, p)$ will be called **lagrangian singularity**. There are several ways of describing a lagrangian subvariety resp. singularity.

Definition 1.1. *Let (M, ω) be symplectic over $\mathbb{K}$. We call an ideal sheaf $\mathcal{I} \subset \mathcal{O}_M$* ***involutive*** *iff it is stable under the Poisson bracket, i.e., iff $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$.*

The following statement, which follows immediately from the definitions relates the algebraic condition of involutiveness of an ideal with the geometry of the subspace that it defines.

Theorem 1.2. *Let $\mathcal{I} \subset \mathcal{O}_M$ be involutive. Then the subspace $L \subset M$ defined by $\mathcal{I}$ is coisotropic on its smooth locus. Moreover, suppose $\mathcal{I}$ to be a radical ideal, which is pure of dimension n, then L is lagrangian. If $\mathcal{I}$ is prime, then L is lagrangian iff $\mathcal{I}$ is maximal (but not equal to $\mathcal{O}_M$) among all involutive ideals.*

In the examples which will be given later, we always consider lagrangian singularities with its reduced structure. A simple but important observation is that involutiveness can be checked on the generators of an ideal.

Lemma 1.3. *Let $I \subset \mathcal{O}_{M,0}$ be generated by $f_1, \ldots, f_k$. Then I is involutive iff $\{f_i, f_j\} \subset I$ for all $i, j \in \{1, \ldots, k\}$.*

This description allows us to check whether a given subspace is lagrangian in a purely algebraic way. As a first (and rather trivial) example, we remark that any curve C in $\mathbb{K}^2$ is a lagrangian subvariety with respect any symplectic structure of $\mathbb{K}^2$ given by a volume form, because $\{f, f\}$ always vanishes. The involutivity of an ideal can be nicely expressed by the so-called structure constants.

Definition 1.4. *Coefficients $A_{ij}^{(k)}$ defined by the expression*

$$\{f_i, f_j\} = \sum_{i=1}^{k} A_{ij}^{(k)} f_k$$

are called structure constants of $f_1, \ldots, f_k$. Note that these functions are not unique.

There is another method of describing a lagrangian singularity, namely, generating families. This notion is used in several ways in the literature, we will describe two different meanings of it. First we recall the well-known principle of *symplectic reduction*, which is used to define generating families and which will appear at several places later. The general situation is the following: Consider a germ $(C, 0) \subset (M, 0)$ of a smooth coisotropic submanifold C of dimension $2n-k$ ($k \in \{1,\dots,n\}$) inside a symplectic manifold (M,ω) of dimension $2n$ and a germ $(M',0)$ of a symplectic manifold (M',ω') of dimension $2(n-k)$ together with a submersion $\pi : C \to M'$ such that $i^*\omega = \pi^*\omega'$ where $i : C \hookrightarrow M$ is the inclusion. M' is the space of integral manifolds of the integrable distribution $(T_pC)^\perp \subset T_pM$.

Theorem 1.5. *Let $(L,0) \subset (M,0)$ be a germ of a smooth lagrangian submanifold L. Suppose that the restriction of the morphism π to $C \cap L$ is finite. Then the germ at zero of the image $L' := \pi(L)$ is analytic in $(M',0)$ and lagrangian with respect to the symplectic form ω'. L' is smooth iff the intersection of L and C is transversal.*

Now suppose that the symplectic manifold is the cotangent bundle. Let $(L,0) \subset (T^*B,0)$ be a lagrangian singularity. Denote by $l : (L,0) \hookrightarrow (T^*B,0) \twoheadrightarrow (B,0)$ the projection on the base. Consider a function germ $f : (X,0) \times (B,0) \to \mathbb{K}$ where X is smooth of dimension m. Suppose that $f_0 : X \to \mathbb{K}$ is a function with isolated critical points. Denote by $\widetilde{L} \subset T^*X \times T^*B$ the image of df. Consider the projection $\pi : X \times T^*B \twoheadrightarrow T^*B$ (note that $X \times T^*B$ is coisotropic in $T^*X \times T^*B$). The restriction of the projection π to the intersection $C := (X \times T^*B) \cap \widetilde{L}$ is finite because C is the critical space of the function f which is already finite over the parameter space B (because f_0 has isolated critical points). Therefore, we can define $Lag(f) \subset T^*B$ to be the reduced lagrangian subvariety, i.e. $Lag(f) := \pi(\widetilde{L} \cap X)$.

Definition 1.6. *We call f a **generating family** for l iff $L = Lag(f)$.*

First note that the constructed $Lag(f)$ is not necessarily singular. It is smooth iff $\widetilde{L}$ and $X \times T^*B$ intersect transversally. This is equivalent to the condition that the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_i \partial q_k} \right)$$

(where $(\mathbf{x},\mathbf{q})$ are coordinates on $X \times B$) has maximal rank at the origin. However, even in this case the projection l needs not to be regular. It is a classical result of Arnold (see [AGZV85]) that germs of lagrangian projections $l : (L,0) \to (B,0)$ (with L smooth) up to symplectomorphisms respecting the bundle structure $T^*B \to B$ are in one to one correspondence with generating families $f : X \times B \to \mathbb{K}$ (where X can vary) up to *stable* $\mathcal{R}^+$*-equivalence*. This allows one to deduce a classification of such projections from the usual classification of functions with isolated critical points.

In general, the space $Lag(f)$ will be singular. We give one example to illustrate the principle of generating functions. Let $X = \mathbb{K}$ and $B = \mathbb{K}^2$. Choose coordinates x on X and p_1, p_2, q_1, q_2 on T^*B. Consider the function $f = x^4 + q_1 x^3 + q_2 x^2$. This is in some sense the simplest example for $\dim(B) = 2$ and $\dim(X) = 1$ as the function $\partial_x \partial_{q_i} F$ must vanish at the origin (for $i = 1, 2$) to give a singular surface. By definition, we have

$$Lag(f) = \left\{ (p_1, p_2, q_1, q_2) \in T^*B \mid \exists x : \frac{\partial f}{\partial x}(x, \mathbf{q}) = 0, p_i = \frac{\partial f}{\partial q_i}(\mathbf{x}, q) \right\}$$

This variety is given by three equations:

$$f_1 := p_2^2 + \tfrac{3}{4} p_1 q_1 + \tfrac{1}{2} p_2 q_2$$

$$f_2 := p_1 q_1^2 + \tfrac{2}{3} p_2 q_1 q_2 - \tfrac{16}{9} p_1 p_2 - \tfrac{8}{9} p_1 q_2$$

$$f_3 := p_1 p_2 q_1 - \tfrac{1}{2} p_1 q_1 q_2 - \tfrac{1}{3} p_2 q_2^2 + \tfrac{4}{3} p_1^2$$

These are the 2×2-minors of the following 3×2-matrix

$$\begin{pmatrix} -p_1 q_1 + \frac{1}{3} q_2^2 & -\frac{2}{3} q_1 q_2 + \frac{16}{9} p_1 \\ p_2 + \frac{1}{2} q_2 & -q_1 \\ \frac{3}{4} p_1 & p_2 \end{pmatrix}$$

which implies that L is a Cohen-Macaulay singularity by the theorem of Hilbert-Burch (see [Eis95]). We get the following structure constants:

$$\begin{aligned} \{f_1, f_2\} &= \tfrac{4}{3} q_1 f_1 - \tfrac{1}{4} f_2 \\ \{f_1, f_3\} &= -\tfrac{4}{3} q_2 f_1 - \tfrac{3}{2} f_3 \\ \{f_2, f_3\} &= -\tfrac{4}{3} q_1 q_2 f_1 - \left(\tfrac{1}{6} q_2 - p_2\right) f_2 - \tfrac{8}{3} q_1 f_3 \end{aligned}$$

The singular locus of L is a line, its reduced structure is given by (q_2, p_2, p_1). The Milnor number of the transversal singularity is 3. This can be seen by comparing the Hilbert polynomials of the jacobian ideal of I, saturated in the origin and its radical. We see that the transversal type is an A_3-singularity. Away from the origin, L is locally a product of this plane curve germ with a line. This is a general fact which will be proved later (see lemma 3.31 on page 84).

For a singular lagrangian subspace of T^*B, there might be no generating family. This happens, e.g., for the open Whitney umbrella in $\mathbb{R}^4$ (the proof uses Maslov classes, see, e.g., [CdV01]). However, there is always a generating family in a somewhat extended sense.

Definition 1.7. *Let $(L, 0) \subset (T^*B, 0)$ be a lagrangian singularity. Then a function germ $f : (X, 0) \times (B, 0) \to \mathbb{K}$ where X is smooth is called a generating family in the generalized sense iff L is a union of components of the lagrangian space $Lag(f)$.*

If we consider lagrangian singularities which have a smooth normalization, then we can always construct generating families with additional components. This construction is due to Zakalyukin (see [Zak90]).

Theorem 1.8. *Let $(L, 0) \subset (T^*B, 0)$ be a lagrangian singularity and let a normalization $n : (X, 0) \to (L, 0) \hookrightarrow (T^*B, 0)$ be given, where X is smooth. Then a generating family $F : (X, 0) \times (B, 0) \to \mathbb{K}$ in the generalized sense exists.*

The proof is based on the following simple observation.

Lemma 1.9. *Let $(Y, 0)$ be a germ of a smooth isotropic submanifold of the standard symplectic space $(\mathbb{K}^{2n}, \omega)$. Then there exists a germ $(\Lambda, 0)$ of a smooth lagrangian manifold L which contains $(Y, 0)$.*

Proof. Let $\Phi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ be an isomorphism such that $V := \Phi(Y)$ is a linear subspace of $\mathbb{K}^{2n}$. Then $\omega' := \Phi^*\omega$ vanishes on V, so that V is an isotropic sub-vector space of the symplectic space $(\mathbb{K}^2, \omega')$. There is a lagrangian sub-vector space $\Lambda' \supset V$ and we define $\Lambda := \Phi^{-1}(\Lambda')$. □

Proof of the theorem. Let $M := T^*B \times T^*X$ be the symplectic product of the two cotangent bundles. The submanifold $C := T^*B \times X$ is coisotropic in M. Define $Y \subset C \subset M$ to be the graph of the map $n : X \to T^*B$. It is obvious to see that $(Y, 0)$ is a germ of a smooth isotropic submanifold of M. Thus we can apply the preceding lemma which yields a germ $(\widetilde{L},0)$ of a smooth lagrangian $\widetilde{L}\subset M$ with $(Y,0)\subset(\widetilde{L},0)$. Now consider the symplectic reduction process in M with respect to the submanifold C. Define $L'\subset T^*B$ to be the reduced lagrangian space. It is clear that set-theoretically $L\subset L'$, then, by the irreducibility of L we get that L is a component of L'.

Consider the lagrangian projection $(\widetilde{L},0)\hookrightarrow(T^*(B\times X),0)\twoheadrightarrow(B\times X,0)=:(B',0)$, note that now the source $\widetilde{L}$ is smooth. By the Arnold correspondence there is a generating family $F:X'\times B'\to\mathbb{K}$. This family can be considered as defined on $(X'\times X)\times B$. Then the generated lagrangian is the above constructed L' which contains L as a component, as required. □

We will give a generating family in this extended sense for the open Whitney umbrella in section 1.3 on page 29.

Quite frequently, one also finds the notion of a generating function associated to a lagrangian singularity in the literature. This is a different object than a generating family as above. To explain it, we first need to recall some facts on differential forms on singularities. This will also be useful in the second chapter. Let for a moment $(X,0)\subset(\mathbb{K}^N,0)$ denote any germ of an analytic subspace. Then we can consider several quotients of the module $\Omega_{\mathbb{K}^N,0}$ of differential forms on $\mathbb{K}^N$. The "largest" one is usually called module of *Kähler*-differentials and defined by

$$\Omega_{X,0}:=\frac{\Omega_{\mathbb{K}^N,0}}{I\Omega_{\mathbb{K}^N,0}+dI}$$

where $I\subset\mathcal{O}_{\mathbb{K}^N,0}$ is the defining ideal. The exterior powers of $\Omega_{X,0}$ together with the induced differential form a complex, usually called the *de Rham complex* of the singularity $(X,0)$. However, for our purpose the complex $\widetilde{\Omega}^\bullet_{X,0}$ defined by $\widetilde{\Omega}^p_{X,0}:=\Omega^p_{X,0}/Tors(\Omega^p_{X,0})$ (where $Tors(\Omega^p_{X,0})$ are the torsion submodules of $\Omega^p_{X,0}$) will be more important. It also appears in [Giv88] and was called $\Omega^\bullet_{Giv}$ in [Her02]. Givental defines it as differential forms on $\mathbb{K}^N$ modulo forms which are zero on the smooth part of X. The module of these forms is obviously a quotient of the module of Kähler forms, that is, there is a sequence

$$0\longrightarrow K\longrightarrow\Omega^p_{X,0}\longrightarrow\Omega^p_{Giv}\longrightarrow 0$$

On the smooth locus, $\Omega^p_{X,0}$ and Ω^p_{Giv} coincide, therefore the kernel is a torsion submodule (here we have to suppose that X is reduced). But any torsion element vanishes on X_{reg} so we have $Tors(\Omega^p_{X,0}) \subset K$ and thus $\Omega^p_{Giv} = \widetilde{\Omega}^p_{X,0}$. The following lemma recalls a well-known fact concerning the cohomology of these two complexes.

Lemma 1.10. *Let $(X,0) \subset (\mathbb{K}^N,0)$ be quasi-homogeneous with positive weights. Then*

1. *The de Rham-complex $\Omega^\bullet_{X,0}$ is acyclic except in degree zero where its cohomology are the constant functions.*
2. *The same is true for the complex $\widetilde{\Omega}^\bullet_{X,0}$, we have: $H^i(X,\widetilde{\Omega}^\bullet_{X,0}) = 0$ for $i > 0$ and $H^0(X,\widetilde{\Omega}^\bullet_{X,0}) = \mathbb{K}$.*

Proof. Denote by E the Euler vector field corresponding to te quasi-homogeneous graduation of $\mathcal{O}_{\mathbb{K}^N,0}$, i.e.

$$E = \sum_{i=1}^{N} \lambda_i x_i \partial_{x_i}$$

where $(x_1,\ldots,x_N)$ are coordinates on $\mathbb{K}^N$ and λ_i are their (positive) weights. The equations of X are quasi-homogeneous, thus there is an induced graduation of $\mathcal{O}_{X,0}$ and of $\Omega^p_{X,0}$. For a form ω, homogeneous with respect to this graduation we get $Lie_E(\omega) = w \cdot \omega$ where w is the weight of ω. On the other hand, suppose that $\omega \in H^p(\Omega^\bullet_{X,0})$ for $p > 0$, then $Lie_E(\omega) = di_E\omega$ so with $\alpha := w^{-1} i_E \omega$ for $w \neq 0$ we get $d\alpha = \omega$ meaning that ω is zero in the cohomology. But the only forms with zero weight are the constant functions on L, this implies that $H^\bullet(\Omega^\bullet_{X,0}) = \mathbb{K}_{X,0}$ proving the first statement. To show the corresponding result for the complex $\widetilde{\Omega}^\bullet_{X,0}$, consider the exact sequence of complexes

$$0 \longrightarrow K^\bullet \longrightarrow \Omega^p_{X,0} \longrightarrow \widetilde{\Omega}^\bullet_{X,0} \longrightarrow 0$$

The only point to verify is that for any vector field $X \in \Theta_{X,0}$, the morphism $i_X : \Omega^p_{X,0} \to \Omega^{p-1}_{X,0}$ maps the kernel complex $K^\bullet$ into itself. But this is obvious, because the kernel consists of the torsion subsheaves of $\Omega^p_{X,0}$ and the interior multiplication i_X is linear over $\mathcal{O}_{X,0}$. $\square$

We will now give the definition and some properties of generating functions as described in [Giv88] (some more details can also be found in [Her02]). Let $(M, 0)$ be a germ of a symplectic manifold (M, ω). Denote by α the Liouville form defined in a neighborhood of the origin. Let $(L, 0) \subset (M, 0)$ be a germ of a lagrangian singularity. Consider the restriction $\alpha \in \widetilde{\Omega}^1_{L,0}$. This form is closed in $\widetilde{\Omega}^1_{L,0}$ (because ω vanishes on L_{reg}), thus defining a class $[\alpha] \in H^1(\widetilde{\Omega}^\bullet_{L,0})$. It is an invariant of the lagrangian singularity and was called its class in [Giv88]. However, α is not exact in general. Nevertheless, there is a Whitney regular stratification of L and α can be integrated along pathes corresponding to this stratification. This yields a continuous function F on L which satisfies $dF = \alpha$ on L_{reg}. Therefore, F is analytic on L_{reg}. By definition, we see that $F \in \mathcal{O}^w_{L,0}$, the weak normalization of L. F is called the generating function of L. An obvious question in the situation is to know whether $F \in \mathcal{O}_{L,0}$. Let us restrict to the complex case in the following. If L is e.g. weakly normal, then F is holomorphic on the whole of L. By definition of the complex $\widetilde{\Omega}^\bullet_{L,0}$, if $H^1(\widetilde{\Omega}^\bullet_{L,0})$ is zero, then $F \in \mathcal{O}_{L,0}$. The problem to find a holomorphic generating function is therefore reduced to determine whether $H^1(\widetilde{\Omega}^\bullet_{L,0})$ vanishes or not. The following conjecture of Givental is an analogue of the famous Arnold conjecture (proved by Gromov) for the local complex analytic case (the assumption $H^n(\widetilde{\Omega}^\bullet_{L,0}) \neq 0$ corresponds to the compactness of the real Lagrangians in the Arnold conjecture).

Conjecture 1.11. *If $H^n(\widetilde{\Omega}^\bullet_{L,0}) \neq 0$, then $H^1(\widetilde{\Omega}^\bullet_{L,0}) \neq 0$ and α is not exact.*

For lagrangian curves, this statement is true, the proof uses the Gauß-Manin connection for hypersurface singularities. On the other hand, for a curve $H^1(\widetilde{\Omega}^\bullet_{L,0}) = 0$ vanishes iff $(L, 0)$ is quasi-homogenous. More precisely, we have that $\dim_{\mathbb{C}} \left(H^n(\widetilde{\Omega}_{X,0}) \right) = \mu - \tau$ for any germ of a hypersurface singularity $(X, 0)$ of dimension n (this is a theorem of K. Saito, see [Sai71]). There is another special case where vanishing of the de Rham-cohomology is known, namely, the case of isolated complete intersections. The following statement is taken from [Gre80].

Theorem 1.12. *Let $(L, 0)$ be a complete intersection with isolated sin-*

gularities. Then

- $H^p(\Omega^\bullet_{L,0}) = 0$ *for* $0 < p < n$.
- $H^p(\widetilde{\Omega}^\bullet_{L,0}) = 0$ *for* $p \neq 0, n$.
- $H^n(\widetilde{\Omega}^\bullet_{L,0}) = 0$ *if* $(L, 0)$ *is quasi-homogenous.*

Related to the above definition of generating functions is the notion of the *front* of a lagrangian singularity. We suppose that the symplectic manifold is a cotangent bundle.

Definition 1.13. *Let* $(L, 0) \subset (T^*B, 0)$ *be a lagrangian singularity. Denote by* $\pi : T^*B \to B$ *the canonical projection and suppose that it defines a finite mapping* $\pi : L \to B$. *Let* F *be a generating function. Then the image* Φ_L *of the mapping* $(\pi, F) : L \to B \times \mathbb{K}$ *(which is also finite) is called the front of* L.

As we have said, F is an element of $\mathcal{O}^w_{L,0}$. In particular, it is contained in the normalization and therefore satisfies an algebraic relation $F^k + a_1 F^{k-1} + \ldots + a_k = 0$ with $a_i \in \mathcal{O}_{L,0}$. $\mathcal{O}_{L,0}$ is a finite ring extension of $\mathcal{O}_{B,0}$ and hence there is also a relation of type $F^m + b_1 F^{m-1} + \ldots + b_m = 0$ with $b_i \in \mathcal{O}_{B,0}$. Then the front is the vanishing locus of the polynomial $z^m + b_1 z^{m-1} + \ldots + b_m = 0$ in $B \times \mathbb{K}$ with coordinates $(q_1, \ldots, q_n, z)$ where $(q_1, \ldots, q_n)$ are the coordinates on the base B. This proves that the front is always an analytic hypersurface in $B \times \mathbb{K}$ regardless whether F lies in $\mathcal{O}_{L,0}$ or not.

We will give one example from [Giv88] with non-analytic generating function. We will come back to lagrangian singularities of this type later. Consider the germ $(C, 0)$ of a plane curve C in $\mathbb{C}^2$ given by the equation $f = x^3 + y^7 + xy^5$. We see $\mathbb{C}^2$ as cotangent bundle of $\mathbb{C}$ by the projection $(x, y) \mapsto x$. $(C, 0)$ is a non quasi-homogenous singularity and $H^1(\widetilde{\Omega}^\bullet)$ is one-dimensional generated by the form $x\,dy$. Therefore, the generating function F is not holomorphic on $(C, 0)$. However, we can consider the pullback $n^*\alpha$ and get a closed (and therefore exact) form on the normalization $\widetilde{C}$. This yields a function $F \in \mathcal{O}_{\widetilde{C}}$. Then the image of the map $(F, x) : \widetilde{C} \to \mathbb{C}^2$ is the front of the lagrangian singularity $(C, 0)$. Moreover, the image of the map $(F, n) : \widetilde{C} \to \mathbb{C}^3$ is a legendrian space curve and the front is the front of this legendrian curve in the classical sense if we consider $(\mathbb{C}^3, 0)$ as (the germ of) the space of contact elements of $\mathbb{C}^2$ with projection $(z, x, y) \to (z, x)$.

1.2 Open Swallowtails

Swallowtails are subspaces of manifolds consisting of polynomials (in one variable) of fixed degree with certain coefficients fixed. Let us start with a simple but important example. Consider the space (denoted by $\mathcal{P}_5$) of polynomials $P \in \mathbb{K}[t]$ of degree five, with fixed leading coefficient and sum of roots equal to zero. Such a polynomial can be represented as

$$P(t) = t^5 + xt^3 + yt^2 + zt + w$$

and choosing coordinates (x, y, z, w), the space $\mathcal{P}_5$ is obviously isomorphic to $\mathbb{K}^4$. Let us the following symplectic form: $\omega = 3dx \wedge dw + dz \wedge dy$. The origin of this form will be explained later in a more general context. Consider the subspace of $\mathcal{P}_5$ which consists of polynomials having a root of multiplicity at least three. Denote this space by Σ_2. A polynomial $P \in \Sigma_2$ can be written as $P(t) = (t-a)^3(t^2 + 3at + b)$, so there is a parameterization of Σ_2 (which is in fact the normalization) given by

$$\begin{array}{rcl} n : \mathbb{K}^2 & \longrightarrow & \mathcal{P}_5 \\ (a, b) & \longmapsto & (b - 6a^2, 8a^3 - 3ab, 3a^2b - 3a^4, -a^3b) \end{array}$$

One can check directly that $n^*\omega = 0$. On the other hand, the image is given by the following three polynomials

$$\begin{array}{rcl} f_1 & = & 15xy^2 - 45x^2z + 100z^2 - 375yw \\ f_2 & = & 27y^3 - 96xyz + 135x^2w - 300zw \\ f_3 & = & 9y^2z - 32xz^2 + 15xyw - 375w^2 \end{array}$$

which are in fact the minors of the matrix

$$\begin{pmatrix} 3w & 9y^2 - 32xz \\ z & -5xy + 125w \\ -3y & 45x^2 - 100z \end{pmatrix}$$

Then one can calculate explicitly the commutators:

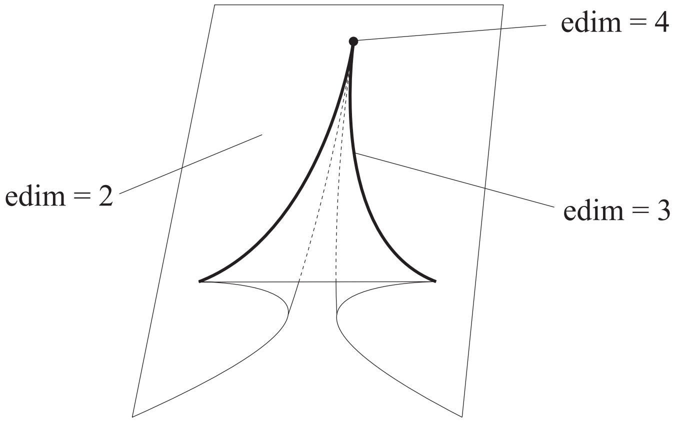


Figure 1.1: The open swallowtail $\Sigma_2 \subset \mathbb{R}^4$

$$\begin{array}{rcl} \{f_1, f_2\} & = & -6xf_1 + 300f_3 \\ \{f_1, f_3\} & = & -4yf_1 - 5xf_2 \\ \{f_2, f_3\} & = & -32zf_1 - 27yf_2 + 192xf_3 \end{array}$$

This shows that $\Sigma_2 \subset \mathcal{P}_5$ is a lagrangian subspace. Its singular locus is a plane curve which has an A_2-singularity at the origin and the transversal singularity is also a plane cusp. The points of $Sing(\Sigma_2)$ correspond to polynomials which have a root of multiplicity four. This can of course be calculated directly, but we will prove it later for general open swallowtails. The only polynomial having a root of multiplicity five in $\mathcal{P}_5$ is t^5. This is the origin in Σ_2. By differentiating an element $P(t)$ in $\mathcal{P}_5$ with respect to t, we obtain a polynomial of degree four with fixed leading coefficient and sum of roots equal to zero. Denote the space of these polynomials by $\mathcal{P}_4$. The subspace Σ_2 is mapped to the space $\Delta_2 \subset \mathcal{P}_4$ of polynomials having a root of multiplicity two. This is a hypersurface in three space, the so-called ordinary swallowtail. It is given in our coordinates by the single equation

$$x^3y^2 + 15y^4 - 3x^4z - 60xy^2z + 40x^2z^2 - 400/3z^3 = 0$$

It has a line of self-intersection. Writing $Q \in \Delta_2$ as $Q(t) = (t-\alpha)^2(t^2 + 2\alpha t + \beta)$ yields a normalization. The self-intersection points are not critical values of this normalization, they correspond to polynomials of type $Q(t) = (t-\alpha)^2(t+\alpha)^2$. These polynomials have two images under the normalization. This phenomenon does not occur for polynomials

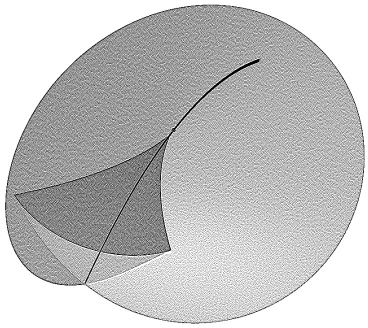


Figure 1.2: The ordinary swallowtail $\Delta_2 \subset \mathbb{R}^3$

in degree five, hence, the line of self-intersection disappears. The ordinary swallowtail is drawn in figure 1.2. Note that over $\mathbb{R}$, the line of self-intersection is continues outside the surface. We will see this phenomenon occurring again in real representations of several other surfaces. A conceptual picture of the open swallowtail is given in figure 1.1 on the page before. We have marked the strata of constant embedding dimension, namely, the regular locus, the smooth points of the singular locus and the origin. Again the variety is a product locally along its singular locus away from the origin. See lemma 3.31 on page 84 and 3.33 on page 85 for further explanations.

The variety Σ_2 is quasi-homogenous with respect to the weights

$$\begin{aligned} \deg(x) = 2, \quad \deg(y) = 3, \\ \deg(z) = 4, \quad \deg(w) = 5 \end{aligned}$$

This implies that for a form $\alpha \in \Omega^1_{\Sigma_2}$ with $d\alpha = \omega$, a generating function $F \in \mathcal{O}_{\Sigma_2}$ exists. For $\alpha = -3wdx + zdy$, we obtain the function $F = 9a^5b - 3a^3b^2 - \frac{72}{7}a^7 \in \widetilde{\mathcal{O}}_{\Sigma_2}$ on the normalization satisfying $dF = n^*\alpha$. Using *Singular* (see [GPS01]), we see that F lies indeed in the subalgebra $\mathcal{O}_{\Sigma_2}$ and can be expressed as $F = \frac{3}{7}yz - \frac{6}{7}xw$. The image of the map $\Sigma_2 \to \mathbb{K}^3$ which sends (x, y, z, w) to $(w, y, F(x, y, z, w))$ is the front of Σ_2. It is the hypersurface given by the following equation, where we take (w, y, t) as coordinates on $\mathbb{K}^3$:

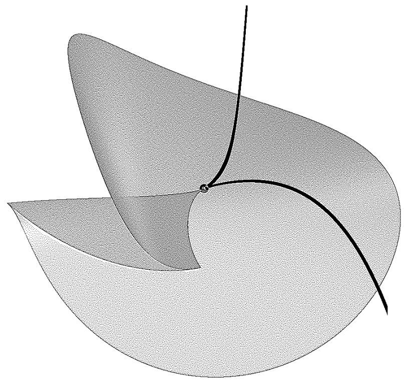


Figure 1.3: The front of the open swallowtail

$$\tfrac{896}{16875}x^6y^3 + \tfrac{203}{625}x^3y^5 - \tfrac{896}{1875}x^7t + y^7$$

$$-\tfrac{2009}{675}x^4y^2t - \tfrac{196}{15}xy^4t + \tfrac{2744}{81}x^2yt^2 - \tfrac{27440}{729}t^3$$

A picture of this surface is given in figure 1.3. Its singular locus is a union of two space curves C_1 and C_2. The transversal type of the front at C_1 is A_4. This is not a surprise: The transversal singularity of the open swallowtail Σ_2 is a cusp, and the front of a cusp is easily seen to be of type A_4. At the other component C_2, the transversal singularity is A_1. This is just a self-intersection of the front, not a singularity of the parameterization. However, over the reals the transversal curve at C_2 is a point, so that the real picture of the front is a union of a surface with a space curve (much like for the ordinary swallowtail in $\mathbb{R}^3$). Note that also C_1 has embedding dimension three, in contrast to the singular locus of $\Sigma_2 \subset \mathbb{K}^4$, which is a plane curve.

In the following definition, we introduce general open swallowtails in polynomial spaces of any (even) dimension.

Definition 1.14. *Denote by* $F_k(x, \mathbf{a}) = x^k + \frac{a_2}{(k-2)!}x^{k-2} + \ldots + a_k$ *the universal unfolding of* x^k*. Let* $\mathcal{P}_k$ *the space of all polynomials* F_k*. In particular, we consider the space of polynomials of odd degree, that is,*

$$\mathcal{P}_{2n+1} = \left\{ x^{2n+1} + \frac{a_2}{(2n-1)!}x^{2n-1} + \ldots + a_{2n+1} \,|\, a_i \in \mathbb{K} \right\} \cong \mathbb{K}^{2n}$$

which comes equipped with the following symplectic structure

$$\omega = \sum_{i=2}^{n+1} (-1)^i da_i \wedge da_{2k+3-i}$$

Let Σ_n be the subspace of polynomials having a root of multiplicity greater than n.

Theorem 1.15. *Consider the open swallowtail $\Sigma_n \subset \mathcal{P}_{2n+1}$.*

1. *Σ_n is lagrangian in $\mathcal{P}_{2n+1}$.*
2. *Σ_n is a Cohen-Macaulay singularity.*

Proof. To prove the first statement, one has to understand the origin of the symplectic structure in $\mathcal{P}_{2n+1}$. This has been done in detail in [Sev99] (and can of course be found in [Giv88]). We only remark that $\mathcal{P}_{2n+1}$ is the two-fold symplectic reduction of the space $\widetilde{\mathcal{P}}_{2n+3}$ of polynomials of degree $2n+3$ without any restriction (this space has dimension $2n+4$). In $\widetilde{\mathcal{P}}_{2n+3}$ one has a natural symplectic structure coming from the representation of $\mathfrak{sl}_2$. By performing only the first symplectic reduction, one obtains an intermediate space $\widehat{\mathcal{P}}_{2n+2}$ of dimension $2n+2$ (which is the space of polynomials of degree $2n+2$ with fixed leading coefficient). Then the second symplectic reduction from $\widehat{\mathcal{P}}_{2n+2}$ onto $\mathcal{P}_{2n+1}$ is in fact the quotient map onto the orbit space of the group action which is the translation of the argument. In $\widehat{\mathcal{P}}_{2n+3}$, the subspace of all polynomials having zero as a root of multiplicity greater than $n+1$ is lagrangian (because half of the coordinates are zero), and by translating the argument one obtains precisely any polynomial having an arbitrary root of multiplicity greater than n.

The second statement is evident for $n = 2$ by the Hilbert-Burch theorem. In higher dimension, we use an argument which can be found in [Giv88]. To prepare it, suppose that for a given singularity $(X, 0)$ we have a finite mapping $(X, 0) \to (Y, 0)$ with Y smooth. Then $\mathcal{O}_{X,0}$ is a Cohen-Macaulay ring if it is a Cohen-Macaulay $\mathcal{O}_{Y,0}$-module. But this is (as $\mathcal{O}_{X,0}$ is $\mathcal{O}_{Y,0}$-finite) equivalent to the condition that $\mathcal{O}_{X,0}$ is a free $\mathcal{O}_{Y,0}$-module. Therefore, to conclude it suffices to prove the following lemma. □

Lemma 1.16. *Let* $\Sigma_n \subset \mathcal{P}_{2n+1}$. *Then*

1. *A normalization of* Σ_n *is given by the following map*

$$\begin{array}{rcl} \varphi : \widetilde{\Sigma}_n := \mathbb{K}^n & \longrightarrow & \Sigma_n \subset \mathcal{P}_{2n+1} \\ (t, a_2, \ldots, a_n) & \longmapsto & (x-t)^{n+1} \cdot (x^n + b_2 x^{n-1} + \ldots + b_n) \end{array}$$

where $b_i \in \mathcal{O}_{\widetilde{\Sigma}_n,0}$ *are chosen such that the coefficient of* t^{2n+1-i} *in the polynomial* $\varphi(t, \mathbf{a})$ *is precisely* $a_i/(2n+1-i)!$ *for* $i = 2, \ldots, n$ *(in particular,* $b_2 = (n+1)t$*).*

2. *We have the following description of* $\mathcal{O}_{\Sigma_n,0}$ *as a subalgebra of* $\mathcal{O}_{\widetilde{\Sigma}_n,0}$*:*

$$\mathcal{O}_{\Sigma_n,0} = \left\{ C(\mathbf{a}) + \int_0^t Q(z, \mathbf{a}) F_n(z, \mathbf{a}) dz \,|\, C \in \mathcal{O}_{\mathcal{P}_{n+1},0}, Q \in \mathcal{O}_{\widetilde{\Sigma}_n,0} \right\}$$

where the function C *of the coordinates* $a_2, \ldots, a_{n+1}$ *is seen as defined on the space of polynomials* $\mathcal{P}_{n+1} = \{x^{n+1} + \frac{a_2}{(n-1)!} x^{n-1} + \ldots + a_{n+1}\}$.

3. *Consider the map* $\mathcal{P}_{2n+1} \to \mathcal{P}_{n+1}$ *given by the* n*-th derivative. Then the restriction* $\Sigma_n \to \mathcal{P}_{n+1}$ *is finite of degree* $n+1$*. Moreover,* $\mathcal{O}_{\Sigma_n,0}$ *is a free* $\mathcal{O}_{\mathcal{P}_{n+1},0}$*-module of rank* $n+1$*.*

Proof. 1. One calculates easily that the b_i's as in the theorem exist and are uniquely defined. Therefore the map φ is well-defined. It is a normalization because for any polynomial $P \in \Sigma_n$, the values t and $a_2, \ldots, a_n$ such that $\varphi(t, \mathbf{a}) = P$ are uniquely determined, so the map is generically one to one.

2. We first show that for any $i = 1, \ldots, n+1$, the following formula holds in the ring $\mathcal{O}_{\widetilde{\Sigma}_n,0}$:

$$a_{n+i} = \frac{(-1)^i}{(i-1)!} \int_0^t F_n(z, \mathbf{a}) z^{i-1} dz$$

Here a_{n+i} is seen as lying in $\mathcal{O}_{\widetilde{\Sigma}_n,0}$ via the inclusion $\varphi^* : \mathcal{O}_{\Sigma_n,0} \hookrightarrow \mathcal{O}_{\widetilde{\Sigma}_n,0}$. We prove this formula by induction on i: let first $i = 1$, then

$$-\int_0^t F_n(z, \mathbf{a}) dz = -F_{n+1}(t, \mathbf{a}) + a_{n+1}$$

But t is a root of $F_{n+1}(z,\mathbf{a})$ (because this is just the n-th derivative of $F_{2n+1}(z,\mathbf{a})$ which is supposed to have a zero of multiplicity $n+1$ at t). For the induction step, we use integration by parts:

$$\tfrac{(-1)^i}{(i-1)!}\int_0^t F_n(z,\mathbf{a})z^{i-1}dz =$$

$$\tfrac{(-1)^i t^i}{(i-1)!}F_{n+1}(t,\mathbf{a}) - \tfrac{(-1)^i}{(i-2)!}\int_0^t z^{i-2}F_{n+1}(z,\mathbf{a})$$

The first term vanishes as above, and by setting $n' := n+1$ we obtain

$$\frac{(-1)^i}{(i-1)!}\int_0^t F_n(z,\mathbf{a})z^{i-1}dz = \frac{(-1)^{i-1}}{(i-2)!}\int_0^t z^{i-2}F_{n'}(z,\mathbf{a}) = a_{n'+(i-1)}$$

by induction hypothesis. But $a_{n'+(i-1)} = a_{n+i}$ so the formula is proved. Using this identity we can already show that any function $g \in \mathcal{O}_{\Sigma_n,0}$ can be represented as required. Lift g to a function $G \in \mathcal{P}_{2n+1,0}$. Then we have

$$\partial_t G = \sum_{i=1}^n \partial_{a_{n+i}} G \cdot \partial_t a_{n+i} = F_n(t,\mathbf{a}) \cdot \left(\sum_{i=1}^n \partial_{a_{n+i}} G \frac{(-1)^i}{(i-1)!} t^{i-1}\right)$$

Thus G has the required form. It remains to show that any function $G = C(\mathbf{a}) + \int_0^t Q(z,\mathbf{a})F_n(z,\mathbf{a})$ can be written as depending only on $a_2,\ldots,a_{2n+1}$, i.e., can be lifted to $\mathcal{O}_{\mathcal{P}_{2n+1},0}$. This will show that functions of this type lie already in $\mathcal{O}_{\Sigma_n,0}$. To do this Givental uses a trick involving a versality theorem for semi-forms. We will not discuss this here in detail but quote the result we need: Any function $\alpha \in \mathbb{K}\{t, a_2, \ldots, a_{n+1}\}$ can be written as

$$\alpha(t,\mathbf{a}) = F_n(t,\mathbf{a})R(t,\mathbf{a}) + \frac{1}{2}F_{n+1}(t,\mathbf{a})\partial_t R(t,\mathbf{a}) + \sum_{i=1}^n \frac{\lambda_i(\mathbf{a})}{(i-1)!}t^{i-1}$$

for functions $R \in \mathcal{O}_{\tilde{\Sigma}_n,0}$ and $\lambda_i \in \mathbb{K}\{\mathbf{a}\}$ (the non-standard term is $\frac{1}{2}F_{n+1}(t,\mathbf{a})\partial_t R(t,\mathbf{a})$). We multiply the above equation by F_{n+1}:

$$\begin{aligned} F_{n+1}(t,\mathbf{a})\alpha(t,\mathbf{a}) &= \tfrac{\partial}{\partial t}\left(R(t,\mathbf{a})\tfrac{F_{n+1}^2(t,\mathbf{a})}{2}\right) \\ &\quad + \textstyle\sum_{i=1}^n \tfrac{\lambda_i(\mathbf{a})}{(i-1)!}t^{i-1}F_{n+1}(t,\mathbf{a}) \end{aligned}$$

and integrate:

$$\begin{aligned}\int_0^t F_{n+1}(z,\mathbf{a})\alpha(z,\mathbf{a})dz &= \sum_{i=1}^{n} \lambda_i(\mathbf{a}) \int_0^t \frac{z^{i-1}F_{n+1}(z,\mathbf{a})}{(i-1)!}dz \\ &- \left(R(0,\mathbf{a})\frac{F_{n+1}^2(0,\mathbf{a})}{2}\right)\end{aligned}$$

Integration by parts yields:

$$\int_0^t F_n(z,\mathbf{a})Q(z,\mathbf{a})dz =$$
$$\sum_{i=1}^{n} \lambda_i(\mathbf{a})a_{n+i+1} - \underbrace{\left(R(0,\mathbf{a})\frac{F_{n+1}^2(0,\mathbf{a})}{2} - F_{n+1}(0,\mathbf{a})Q(0,\mathbf{a})\right)}_{\lambda_0(a_2,\ldots,a_{n+1})}$$

where Q is a primitive of α. Note that we have used two times the fact that t is a root of F_{k+1}. So we have a lift of functions of type $\int_0^t F_n(z,\mathbf{a})Q(z,\mathbf{a})dz + C(\mathbf{a})$ to $\mathcal{O}_{\mathcal{P}_{2n+1},0}$ as required.

3. The map $\Sigma_{n,0} \to \mathcal{P}_{\mathcal{P}_{n+1},0}$ is of degree $n+1$ because any (generic) polynomial with (simple) roots $t_1, \ldots, t_{n+1}$ has $n+1$ preimages under this map, namely, the polynomials $(x-t_j)^{n+1}\prod_{i=1, i\neq j}^{n+1}(x-t_i)$ for $j = 1, \ldots, n+1$. This implies that $\mathcal{O}_{\Sigma_n,0}$ is a finitely generated $\mathcal{O}_{\mathcal{P}_{n+1},0}$-module of rank $n+1$. The last formula shows that it is generated by $1, a_{n+2}, \ldots, a_{2n+1}$, so it must be free.

□

1.3 Conormal cones

Conormal cones are a systematic way to construct lagrangian singularities from given singularities of lower dimension. We first illustrate this with a simple example. Let $(C,0) \subset (\mathbb{K}^2,0)$ be the ordinary cusp singularity, i.e., the germ at zero of the vanishing locus of the polynomial $z^3 - w^2$. Consider the normalization $m : (\mathbb{K},0) \to (C,0)$ given by $s \mapsto (s^2, s^3) = (z,w)$. A vector (a,b) is a normal vector to a point

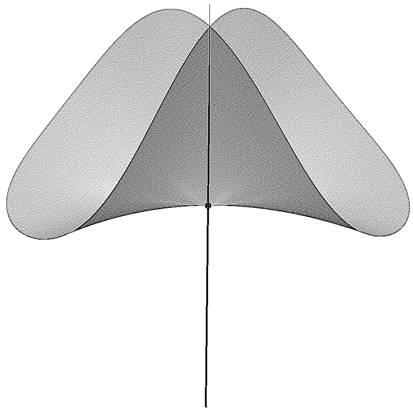


Figure 1.4: The ordinary Whitney umbrella in $\mathbb{R}^3$

$p = m(s) \in C$ iff $2as + 3bs^2 = 0$, or $a = -\frac{3}{2}bs$. Therefore, if we identify the tangent bundle of $\mathbb{K}^2$ with $\mathbb{K}^4$ the map (let (x, y, z, w) be the coordinates in $\mathbb{K}^4$)

$$\begin{array}{rcl} \widetilde{n} : \mathbb{K}^2\backslash(0,0) & \longrightarrow & \mathbb{K}^4 \\ (s,t) & \longmapsto & (-3st, 2t, s^2, s^3) \end{array}$$

is a parameterization of the normal bundle of the smooth part of C. Using the standard metric on $\mathbb{K}^4 = T\mathbb{K}^2$, we can identify tangent and cotangent bundle to obtain a smooth subvariety $\mathcal{W}_2^0$ in the cotangent bundle. $\mathcal{W}_2^0$ is of course just the total space of the *conormal bundle* of C_{reg}. We define $\mathcal{W}_2$ to be the algebraic closure of $\mathcal{W}_0^2$. The projection of $\mathbb{K}^4$ onto $\mathbb{K}^3$ along the w-axis sends $\mathcal{W}_2$ to the so called *ordinary Whitney umbrella* (one also finds the name D_∞-singularity). This surface in three-space is given by the single equation $y^2z - \frac{4}{9}x^2$. It is drawn in figure 1.4. The singular locus of the ordinary Whitney umbrella is a line, whereas $\mathcal{W}_2$ has a unique singular point at the origin. One can think of $\mathcal{W}_2$ as being obtained from the ordinary Whitney umbrella be *unfolding* the singular line. Therefore it was called open (unfolded, unfurled) Whitney umbrella by Givental ([Giv86]). In our example $\mathcal{W}_2$ is given by the following four polynomials.

$$f_1 := xz + \tfrac{3}{2}yw \quad ; \quad f_2 := x^2 - \tfrac{9}{4}y^2z$$

$$f_3 := yz^2 + \tfrac{2}{3}xw \quad ; \quad f_4 := z^3 - w^2$$

Let the symplectic form ω be $dx \wedge dz + dy \wedge dw$. Then the commutators of the above equations are:

$$\{f_1, f_2\} = -2f_2 \quad ; \quad \{f_1, f_3\} = \tfrac{1}{2} f_3$$

$$\{f_1, f_4\} = 3f_4 \quad ; \quad \{f_2, f_3\} = y f_1$$

$$\{f_2, f_4\} = 6z f_1 \quad ; \quad \{f_3, f_4\} = 0$$

This proves that $\mathcal{W}_2$ is lagrangian. By looking at theses commutators, one sees that there are several subsets of $\{f_1, \ldots, f_4\}$ generating ideals which are closed under the Poisson bracket (closed Lie subalgebras). These correspond to lagrangian varieties including $\mathcal{W}_2$ as a component.

closed Lie subalgebra	**ideal of additional component**
(f_1, f_2, f_3)	(x, y)
(f_1, f_2, f_4)	(y^2, w, xz, x^2, z^3)
(f_1, f_3, f_4)	(z, w)
(f_1, f_2)	$(y^2, xy, xz + 3/2yw, x^2)$
(f_1, f_3)	$(x, y) \cap (z, w)$
(f_1, f_4)	$(z^2, xz + 3/2yw, zw, w^2)$
(f_3, f_4)	$(zw, yz^2 + 2/3xw, w^2, z^3)$

Note, however, that only the ideals (f_1, f_2, f_3), (f_1, f_3, f_4) and (f_1, f_3) define spaces with reduced structures. In all cases we get a union of $\mathcal{W}_2$ together with one or two planes (which might have a multiple structure). We have seen that $\mathcal{W}_2$ is not a complete intersection. It is not even a Cohen-Macaulay singularity, because this would force $\mathcal{W}_2$ to be normal (since it is regular in codimension one), but the map $\widetilde{n}$ is in fact a normalization.

The natural projection $(x, y, z, w) \to (z, w)$ is not finite on $\mathcal{W}_2$. Hence there is no front of $\mathcal{W}_2$ with respect to this cotangent fibration. However, the projection $(x, y, z, w) \to (z, y)$ induces a finite map $\mathcal{W}_2 \to \mathbb{K}^2$. The generating function with respect to this projection is $F = -4yw$ and the associated front in $\mathbb{K}^3$ is given by the equation $x^2y^3 - z^2$ (see picture 1.5 on the following page). This surface is called *composed* Whitney umbrella in [Giv86]. We will encounter the open Whitney umbrella,

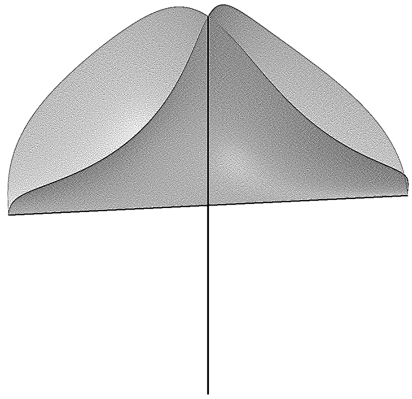


Figure 1.5: The front of the open Whitney umbrella

embedded in this cotangent fibration once again in the last chapter (definition 4.7 on page 119).

The construction of the open Whitney umbrella from a plane cusp can of course be done in much greater generality. More precisely, let X be a smooth N-dimensional manifold. Let T^*X be the cotangent bundle of X and Y a smooth submanifold of X. Then the conormal bundle of Y in X is defined as

$$T^*_Y X := \left\{ \lambda \in T^*X_{|Y} \mid \lambda_{|TY} \equiv 0 \right\} \subset T^*X_{|Y} \subset T^*X$$

By choosing local coordinates, one sees immediately that the total space of $T^*_Y X$ is always a lagrangian submanifold of the symplectic manifold T^*X, regardless of what the dimension of Y is (extreme cases are: $Y = X$ then $T^*_Y X$ is the zero section of T^*X and $Y = \{pt\}$ then its conormal bundle is just the fibre of the fibration $T^*X \to X$ over the point Y). Now suppose that we are given an arbitrary (not necessarily smooth) reduced analytic subspace $Y \subset X$. Define

$$C^*_Y X := \overline{\left\{ \lambda \in T^*X_{|Y_{reg}} \mid \lambda_{|TY} \equiv 0 \right\}}$$

Lemma 1.17. *$C^*_Y X$ (which is also denoted by $T^*_Y X$) is a lagrangian subvariety of the cotangent bundle. It is a conical variety in the fibre direction of T^*X, that is*

$$(p, q) \in C^*_Y X \iff (\lambda p, q) \in C^*_Y X \qquad \forall \lambda \in \mathbb{K}^*$$

Proof. The conormals to smooth points are dense in their closure, so a dense subset of $C^*_Y X$ is lagrangian, meaning that the whole space is a lagrangian subvariety. $C^*_Y X$ is obviously conical, as the vanishing of a form is equivalent to the vanishing of a non-zero multiple of it. □

Characteristic varieties of holonomic $\mathcal{D}$-modules are unions of conormal cones. We explain the relevant notions in some detail in Appendix B, see in particular lemma B.8 on page 178. In the following theorem, taken from [Giv88], generalized Whitney umbrellas in any even dimension are introduced.

Theorem+Definition 1.18. *Define the open Whitney umbrella $\mathcal{W}_{2n}$ by one of the following equivalent descriptions.*

1. *$\mathcal{W}_{2n} \subset \mathbb{K}^{4n}$ is the conormal cone to the open swallowtail $\Sigma_n \subset \mathbb{K}^{2n}$ (see section 1.2 on page 22).*
2. *$\mathcal{W}_{2n}$ is the submanifold of the space of pairs of polynomials of type*

$$F = \tfrac{z^{2n+1}}{(2n+1)!} + a_1 \tfrac{z^{2n-1}}{(2n-1)!} + \ldots + a_{2n}$$

$$G = (-1)^{2n} b_{2n} \tfrac{z^{2n-1}}{(2n-1)!} + b_{2n-1} \tfrac{z^{2n-2}}{(2n-2)!} + \ldots + b_1$$

consisting of (F, G) with a common root t of multiplicity $(n+1, n)$.

3. *Let*

$$\begin{array}{rcl} F_n(\mathbf{q}, \mathbf{Q}, t) & = & \int_0^t \left(Q_1 z^{n-1} + \ldots + Q_n\right) \\ & & \cdot \left(z^{n+1} + q_1 z^{n-1} + \ldots + q_n\right) dz \end{array}$$

Then F_n is a generating family in the generalized sense of $\mathcal{W}_{2n}$.

Proof. We will first show the equivalence of the first two definitions. Consider the following parameterization of the open swallowtail (note that this is not the same as in section 1.2).

$$\begin{array}{rcl} n : \mathbb{K}^n & \longrightarrow & \mathcal{P}_{2n+1} \\ (q_1, \ldots, q_{n-1}, t) & \longmapsto & (z-t)^{n+1} \cdot \\ & & (z^n + (n+1) t z^{n-1} + q_1 z^{n-2} + \ldots + q_{n-1}) \end{array}$$

The derivative Dn of n, restricted to the regular locus of n is an isomorphism from the total space of the tangent bundle of $\mathbb{K}^n$ (that is, from $\mathbb{K}^{2n}$) to the tangent bundle of $(\mathcal{W}_{2n})_{reg}$. The closure of the latter equals the conormal cone $C^*_{\mathcal{W}_{2n}}\mathcal{P}_{2n+1}$ (because Σ_n is lagrangian in $\mathcal{P}_{2n+1}$). But the image of $Dn(\mathbf{q}, t)$ (the tangent space of $n(\mathbf{q}, t)$) consists of all polynomials of degree $2n-1$ with t a root of multiplicity at least n.

Now we show that one component of the variety generated by the family F_n equals $\mathcal{W}_{2n}$. The equation $\partial_t F_n = 0$ is a product, the component describing $\mathcal{W}_{2n}$ is $t^{n+1} + q_1 t^{n-1} + \ldots + q_n$. Consider $p_i := \partial_{q_i} F_n$ and $P_i := \partial_{Q_i} F_n$. It follows easily from lemma 1.16 on page 27 that the map $(t, q_1, \ldots, q_{n-1}) \mapsto (P_1, \ldots, P_n, q_1, \ldots, q_n)$ is the normalization of the n-dimensional swallowtail, i.e., the image of a point $(t, \mathbf{q})$ is a polynomial of degree $2n+1$ with t a root of multiplicity $n+1$. For this t, the image of the map $(Q_1, \ldots, Q_{n-1}, t) \mapsto (p_1, \ldots, p_n, Q_1, \ldots, Q_n)$ is a polynomial of degree $2n-1$ with t a root of multiplicity $n-1$. Therefore, the map $(t, q_1, \ldots, q_{n-1}, Q_1, \ldots, Q_n) \mapsto (\mathbf{P}, \mathbf{p}, \mathbf{Q}, \mathbf{q})$ is a normalization of $\mathcal{W}_{2n}$. □

In [Giv88], there is yet another characterization of $\mathcal{W}_{2n}$. We give it here without proof. Denote by $\widetilde{\mathcal{W}}_{2n}$ the normalization of $\mathcal{W}_{2n}$. Consider the so-called Morin map (see [Mor65]):

$$\begin{array}{rcl} \widetilde{\mathcal{W}}_{2n} & \longrightarrow & \mathbb{K}^{2n+1} \\ (Q_1, \ldots, Q_n, q_1, \ldots, q_{n-1}, t) & \longmapsto & (Q_1, \ldots, Q_n, q_1, \ldots, q_n, p_n) \end{array}$$

It can be seen as the restriction of the projection

$$\begin{array}{rcl} \mathbb{K}^{2n+2} & \longrightarrow & \mathbb{K}^{2n+1} \\ (Q_1, \ldots, Q_n, q_1, \ldots, q_n, p_n, t) & \longmapsto & (Q_1, \ldots, Q_n, q_1, \ldots, q_n, p_n) \end{array}$$

to the codimension two submanifold given by $F = t^{n+1} + q_1 t^{n-1} + \ldots + q_n t + q_n$ and $G = Q_1 t^n + Q_1 t^{n-1} + \ldots + Q_n t + p_n$. Let $K \subset \Theta_{\widetilde{\mathcal{W}}_{2n},0}$ be the kernel of the derivative of the Morin map at zero. Then there is the following equality of subalgebras of $\mathcal{O}_{\widetilde{\mathcal{W}}_{2n},0}$

$$\mathcal{O}_{\mathcal{W}_{2n},0} = \left\{ f \in \mathcal{O}_{\widetilde{\mathcal{W}}_{2n},0} \mid K(f) \in \mathbf{m}_{\mathcal{O}_{\widetilde{\mathcal{W}}_{2n}}} \right\}$$

Of course the definition of the open Whitney umbrella as conormal cone of the open swallowtail applies to our first example: Σ_1 is just the ordinary cusp in the plane, its conormal space is the two-dimensional open Whitney umbrella $\mathcal{W}_2$.

1.4 Integrable systems

A very important class of lagrangian singularities arises when one supposes that an involutive ideal $\mathcal{I}$ is generated by exactly n equations $f_1, \ldots, f_n$ (i.e., the lagrangian singularity is a complete intersection) such that the Poisson brackets of these generators are zero not only in $\mathcal{O}_L$ but on the whole of M. Then the map $F = (f_1, \ldots, f_n) : M \to \mathbb{K}^n$, all fibres of which are lagrangian subspaces of M, is called a (completely) integrable system. The simplest integrable system is again a curve in the plane (the case $n = 1$): the Poisson bracket of its defining equation with itself vanishes. The next step is to consider products of such curves: In general, given two lagrangian subvarieties $L_1 \subset M_1$ and $L_2 \subset M_2$, the product $L_1 \times L_2$ is lagrangian in the symplectic product $(M_1 \times M_2, pr_1^*\omega_1 - pr_2^*\omega_2)$, pr_i being the projections. If we take n curves $C_i \subset M_i \cong \mathbb{K}^2$ with defining equations $f_i \in \mathbb{K}\{p_i, q_i\}$, then $C_1 \times \ldots \times C_n$ is lagrangian in $\prod_{i=1}^n M_i \cong \mathbb{K}^{2n}$ and the system $(f_1, \ldots, f_n)$ is integrable. As an example, consider the product of two cusps given by $f_1 = x^2 - y^3$ and $f_2 = s^2 - t^3$ in four-space. This is a lagrangian surface with one dimensional singular locus which consists of two components isomorphic to the two cusps. The transversal singularity at a singular point obviously is also a cusp.

In order to get more interesting examples, we use the following trick: Consider the case $n = 2$, choose coordinates (p_1, q_1, p_2, q_2) of $\mathbb{K}^4$ and set $z_1 = p_1 + iq_1$ and $z_2 = p_2 + iq_2$ (This can obviously be done only in the real case, but it is a formal calculus which works as well for $\mathbb{K} = \mathbb{C}$ as for $\mathbb{K} = \mathbb{R}$). We can now express functions on $\mathbb{K}^4$ in the variables $z_1, z_2, \overline{z_1}, \overline{z_2}$, and the Poisson bracket becomes

$$\{f, g\} = 2i \left(\partial_{\overline{z}_1} f \cdot \partial_{z_1} g - \partial_{\overline{z}_1} g \cdot \partial_{z_1} f + \partial_{\overline{z}_2} f \cdot \partial_{z_2} g - \partial_{\overline{z}_2} g \cdot \partial_{z_2} f\right)$$

We want to find functions f_1, f_2 such that $\{f_1, f_2\} = 0$. Set, for example $f = \lambda z_1 \overline{z_1} + \mu z_2 \overline{z_2}$ and let us look for a $g = z_1^\alpha \overline{z_1}^\beta z_2^\gamma \overline{z_2}^\delta$ for some parameters $\lambda, \mu, \alpha, \beta, \gamma, \delta \in \mathbb{N}$. It can easily be verified that the commuting condition transforms to $\lambda(\alpha-\beta)-\mu(\gamma-\delta)=0$. The following table shows the equations for some coefficients λ, μ and exponents $\alpha, \beta, \gamma, \delta$.

λ, μ	$\alpha, \beta, \gamma, \delta$	**equations**
$1,0$	$0,0,1,1$	$p_1^2+q_1^2, p_2^2+q_2^2$
$1,2$	$0,2,1,0$	$p_1^2+q_1^2+2(p_2^2+q_2^2), p_2(p_1^2-q_1^2)+2p_1q_1q_2$
$1,3$	$3,0,0,1$	$p_1^2+q_1^2+3p_2^2+3q_2^2, 6q_2p_1^2q_1-2q_2q_1^3+2p_2p_1^3-6p_2p_1q_1^2$
$1,4$	$4,0,0,1$	$p_1^2+q_1^2+4p_2^2+4q_2^2,$ $2p_1^4p_2+8p_1^3q_1q_2-12p_1^2q_1^2p_2-8p_1q_1^3q_2+2q_1^4p_2$
$1,2$	$1,3,1,0$	$p_1^2+q_1^2+2(p_2^2+q_2^2), 2p_1^4p_2+4p_1^3q_1q_2+4p_1q_1^3q_2-2q_1^4p_2$
$2,3$	$3,0,0,2$	$2p_1^2+2q_1^2+3p_2^2+3q_2^2,$ $2p_1^3p_2^2-2p_1^3q_2^2+12p_1^2q_1p_2q_2-6p_1q_1^2p_2^2+6p_1q_1^2q_2^2-4q_1^3p_2q_2$
$2,5$	$5,0,0,2$	$2p_1^2+5p_2^2+2q_1^2+5q_2^2, p_1^5p_2^2-10p_1^3p_2^2q_1^2+5p_1p_2^2q_1^4+10p_1^4p_2q_1q_2$ $-20p_1^2p_2q_1^3q_2+2p_2q_1^5q_2-p_1^5q_2^2+10p_1^3q_1^2q_2^2-5p_1q_1^4q_2^2$

Remark that only in the first four cases we obtain reduced structures. It is of course always possible to calculate with the radicals, but they are in general no longer complete intersections.

One might ask whether there are complete intersection singularities whose defining ideal does not admit a commuting system of generators (see also [CdV01]). As there seems to be no such example, we state the following conjecture.

Conjecture 1.19. *Let $(L,0)\subset(\mathbb{K}^{2n},0)$ be a lagrangian singularity which is a complete intersection. Then L defines an integrable system, i.e., there is a set of generators $f_1,\ldots,f_n$ of the ideal $I\subset\mathcal{O}_{M,0}$ defining L in M such that $\{f_i,f_j\}=0$ in $\mathcal{O}_{M,0}$.*

1.5 The $\mu/2$-stratum

We will encounter the open swallowtail once again in this section. Surprisingly enough, it appears in a different space with different symplectic structure. The mapping sending the swallowtail as defined before to the "new" one turns out to carry one symplectic structure into the other.

We start, as in section 1.2 on page 22 with the space of polynomials

$$\mathcal{P}_5 = \{t^5 + xt^3 + yt^2 + zt + w\}$$

together with the symplectic structure $\omega = dx \wedge dw + 3dz \wedge dy + xdx \wedge dy$. Now consider the subspace of polynomials having two roots, each of multiplicity two. Like before, any such polynomial can be written as $Q = (t-a)^2(t-b)^2(t+2a+2b)$ yielding a normalization

$$\begin{array}{rcll} n : \mathbb{K}^2 & \longrightarrow & \mathcal{P}_5 & \\ (a,b) & \longmapsto & \Big(-3a^2 - 3b^2 - 4ab & , \quad 2a^3 + 2b^3 + 8a^2b + 8ab^2, \\ & & -7a^2b^2 - 4a^3b - 4ab^3 & , \quad 2a^3b^2 + 2a^2b^3 \Big) \end{array}$$

One obtains again a determinantal variety in $\mathbb{K}^4$, which we denote by B^2, where 2 stands for the number of double roots of the polynomials that are the points of B^2. Define the following map

$$\begin{array}{rcl} R : \mathcal{P}_5 & \longrightarrow & \mathcal{P}_5 \\ (x,y,z,w) & \longmapsto & \left(\frac{3}{2}x, 3y, 3x^2 - 12z, 8w - \frac{1}{2}xy\right) \end{array}$$

It can be checked by an explicit calculation that R is an automorphism of $\mathcal{P}_5$ which sends B^2 to Σ_2 and which interchanges (up to a factor) the two symplectic structures.

As before, we consider the spaces $\mathcal{P}_{2n+1}$ for any n. Let $B^n \subset \mathcal{P}_{2n+1}$ be the space of all polynomials having n roots of multiplicity two. Then we have the following

Theorem 1.20. *Consider the space $\overline{\mathcal{P}}_{2n+1}$ of polynomials of degree $2n+1$ with arbitrary sum of roots, i.e., the space of polynomials of type $P(t) = t^{2n+1} + a_0t^{2n} + \ldots + a_{2n}$. This space is canonically graded by setting $deg(a_i) = i$. Define the following map*

$$\begin{array}{rcl} R : \overline{\mathcal{P}}_{2n+1} & \longrightarrow & \overline{\mathcal{P}}_{2n+1} \\ P(t) & \longmapsto & R(P)(x) \end{array}$$

where the polynomial $R(P)(x)$ is defined as

$$R(P)(x) := Res_{t=\infty} \left(t^{2n} \left(1 - \frac{x}{t}\right)^{n-\frac{1}{2}} \left(1 + \frac{a_0}{t} + \ldots + \frac{a_{2n}}{t^{2n+1}}\right)^{\frac{1}{2}} \right)$$

The map R is an automorphism of the space $\overline{\mathcal{P}}_{2n+1}$. It sends the subspace $\mathcal{P}_{2n+1}$ into itself (thus defining an automorphism of $\mathcal{P}_{2n+1}$) and the subspace $B^n \subset \mathcal{P}_{2n+1}$ of polynomials having n double roots to the space $\Sigma_n \subset \mathcal{P}_{2n+1}$ of polynomials having one root of multiplicity $n+1$. The space B^n is lagrangian with respect to the symplectic form $R^*\omega$ (where ω is the natural symplectic structure in $\mathcal{P}_{2n+1}$ constructed above).

Proof. We use a Taylor expansion. One finds that

$$(1-p)^{n-\frac{1}{2}} = 1 - \left(n - \tfrac{1}{2}\right) p + \left(n - \tfrac{1}{2}\right)\left(n - \tfrac{3}{2}\right) p^2 - \dots$$

$$(1+q)^{\frac{1}{2}} = 1 + \tfrac{1}{2}q - \tfrac{1}{8}q^2 + \dots$$

We substitute the above expressions and compute first modulo the ideal $(a_0, \dots, a_{2n})^2$ to obtain

$$R(P)(x) = Res_{t=\infty}\Bigg(\left(1 - \tfrac{2n-1}{2}\tfrac{x}{t} + \tfrac{2n-1}{2}\tfrac{2n-3}{2}\left(\tfrac{x}{t}\right)^2 + \dots\right) \cdot \left(t^{2n} + \tfrac{a_0}{2}t^{2n-1} + \dots + \tfrac{a_{2n}}{2}t^{-1}\right)\Bigg) \quad \mathbf{mod}\ \mathbf{a}^2$$

The first factor does not contain any a_i and all coefficients are non-zero. Therefore, the polynomial $R(P)$ has a fixed highest order coefficient, i.e., the map R is well-defined. Moreover, R is invertible and respects the grading. This implies that if the coefficient a_0 vanishes, then the sum of roots of $R(P)$ also vanishes. Therefore we get an automorphism of $\mathcal{P}_{2n+1}$.

Now we prove that R sends B^n to Σ_n. Any $P \in B^n$ can be written as $P(t) = (t-a)\prod_{i=1}^{n}(t-\lambda_i)^2$. Then we have

$$R(P)(x) = Res_{t=\infty}\left(\sqrt{(t-a)(t-x)^{2n-1}}\prod_{i=1}^{n}(t-\lambda_i)\right)$$

and moreover

$$R(P)^{(k)}(x) = c_k \cdot Res_{t=\infty}\left(\sqrt{(t-a)}(t-x)^{n-k-\frac{1}{2}}\prod_{i=1}^{n}(t-\lambda_i)\right)$$

where c_k is the constant factor $(-1)^k \frac{(2n-1)\cdot(2n-3)\cdot\ldots\cdot(2n-2k+1)}{2^k}$. This shows that the expression under *Res* is regular at infinity for $x = a$ and $k \leq n$. In other words, $R(P)^{(k)}(a) = 0$ for $k = 0, \ldots, n$, which proves that $R(P) \in \Sigma_n$.

The proof of the last statement (the fact that B^n is lagrangian with respect to $R^*\omega$) will be postponed after we have introduced the symplectic structure $R^*\omega$ in a canonical way. □

The space $\mathcal{P}_{2n+1}$ can of course be seen as the universal unfolding of the A_{2n}-singularity. We will introduce a canonical symplectic structure on the unfolding space of any function with isolated critical points. Our main reference for the following paragraphs is [VG82].

Consider the germ of a holomorphic function

$$f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$$

with *isolated critical points*. This amounts to say that the *Milnor algebra* $\mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$ (where J_f is the Jacobi ideal of f) is finite dimensional over $\mathbb{C}$ (denote its dimension by μ). Then it is well known that a *semi-universal* unfolding of f is given by a germ of a function

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu}, 0) \longrightarrow (\mathbb{C}, 0)$$

with $F(\mathbf{x}, \mathbf{t}) = f(\mathbf{x}) + \sum_{i=0}^{\mu} g_i \cdot t_i$, where $g_1, \ldots, g_\mu$ is a chosen basis of the Milnor algebra. Moreover, it is possible and often convenient to take $g_1 = 1$. Following standard terminology, we will also call the morphism

$$\begin{array}{rcl} \varphi : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu}, 0) & \longrightarrow & (\mathbb{C} \times \mathbb{C}^{\mu}, 0) \\ (\mathbf{x}, \mathbf{t}) & \longmapsto & (F(\mathbf{x}, \mathbf{t}), \mathbf{t}) \end{array}$$

an unfolding of f. We need to choose representatives of these germs, they have to respect certain (transversality) conditions. The existence of good representatives follows from general results as found, e.g. in [Loo84]. Denote by $M \subset \mathbb{C}^{\mu}$, $S \subset \mathbb{C}$ resp. $X \subset \mathbb{C}^{n+1}$ small neighborhoods of 0 in $\mathbb{C}^{\mu}$, $\mathbb{C}$ resp. $\mathbb{C}^{n+1}$ such that $F : X \times M \to S$ and $\varphi : X \times M \to S \times M$ are representatives of the above germs with the desired properties. There are distinguished hypersurfaces of M (discussed in [Her02]), namely, the discriminant, the caustic and the bifurcation diagram. We are only interested in the discriminant here. There are several ways to introduce it: We first define the critical space of the unfolding F to be

$$C_F := \{(\mathbf{x}, \mathbf{t}) \in X \times M \mid d_x F(\mathbf{x}, \mathbf{t}) = 0\}$$

The complex structure of C_F is taken to be the one given by the Jacobi ideal $(\partial_{x_i} F)$. It will be in general non-reduced. One might define the "big discriminant" as $\check{D} := \varphi(C_F) \subset S \times M$ and the discriminant as $D := \varphi(C \cap F^{-1}(0)) \subset \{0\} \times M \cong M$. It is the hypersurface of parameters t such that the deformed *singularity*, that is, the zero fibre of the deformed function $F_{\mathbf{t}}$ is still singular. An important fact is that the regular locus D_{reg} consists of those parameters t where $F_t^{-1}(0)$ has exactly one double point (an A_1-singularity). Consider the hypersurface $V := F^{-1}(0) \subset X \times M$. Then the restriction of the projection $X \times M \to M$ to $V \cap F^{-1}(M\backslash D)$ is a smooth morphism whose fibres are all homotopy equivalent to the Milnor fibre of original function f. Therefore, we have a well-defined holomorphic vector bundle $H \to M\backslash D$ of rank μ whose fibres over a point $t \in M\backslash D$ are the cohomology spaces $H^n(V_t, \mathbb{C}) = H^n(\varphi^{-1}(0,t), \mathbb{C})$. This bundle comes with a flat structure, defining the *Gauß-Manin connection* ∇ on H. Denote by $\mathcal{H}$ the sheaf of holomorphic sections of H. Then one might ask about possible extensions of $\mathcal{H}$ over the discriminant D. The second part of [Her02] contains an extensive study of this problem. We quote one result.

Theorem 1.21. *Denote by $i : M\backslash D \hookrightarrow M$ the inclusion. Let $k \in \mathbb{Z}$ be fixed. Then there is a coherent sheaf $\mathcal{H}^{(k)}$ of $\mathcal{O}_M$-modules, which is a subsheaf of $i_*\mathcal{H}$ with the following properties: There is a connection ∇ on $\mathcal{H}^{(k)}$, meromorphic along D, i.e., a morphism*

$$\nabla : \mathcal{H}^{(k)} \longrightarrow \mathcal{H}^{(k)} \otimes \Omega_M(*D)$$

which is logarithmic (meaning that the image of ∇ is contained in $\mathcal{H}^{(k)} \otimes \Omega_M(log\ D)$). Moreover, the ***residue endomorphism*** *of ∇ along D_{reg} (see [Her02], chapter 8, for a precise definition) is*

- *semi-simple with eigenvalues $\frac{n-1}{2} - k$ (with multiplicity one) and zero (with multiplicity $\mu - 1$) in case that $\frac{n-1}{2} \neq k$*

- *nilpotent with one Jordan block of size two in case that* $\frac{n-1}{2} = k$

These sheaves form a good filtration (definition B.4 on page 176) on the *Gauß-Manin system* (see, e.g., [Oda87] and the references therein).

Now that we know about the existence of the modules $\mathcal{H}^{(k)}$ we describe how to construct sections of it. Consider the sheaf of differential n-forms $\Omega^n_{X\times M}$. For any form $\omega \in \Omega^n_{X\times M}$, the restriction to V_t for $t \notin D$ is closed and defines an element of $H^n(V_t, \mathbb{C})$. Thus the map $P_\omega : t \mapsto [\omega]_t \in H^n(V_t, \mathbb{C})$ is a well-defined section of the bundle $\mathcal{H}$. We can also see it as an element of $i_*\mathcal{H}$. Then we have the following.

Lemma 1.22. *The section P_ω lies in $\mathcal{H}^{(-1)}$.*

Proof. The first case to consider is that of a non-degenerate critical point. Its Milnor number equals one, thus there is only one vanishing cycle γ. Let t be the coordinate on M (which is also one-dimensional). It is classical to prove (see [AGZV88] or [Arn90]) that

$$\int_\gamma \omega = ct^{\frac{n+1}{2}} + \dots$$

with $c \neq 0$ and where the dots stand for higher order terms. Hence, for a general function, the residue endomorphism along D_{reg} has $\frac{n+1}{2}$ as an eigenvalue proving that $P_\omega \in \mathcal{H}^{(-1)}$. □

Following Varchenko and Givental, we will call the map P_ω a period map (in a similar situation, such a map is called infinitesimal period map in [Sab02]). Any period map defines via the Gauß-Manin connection a morphism from the tangent bundle to $\mathcal{H}^{(-1)}$, namely:

$$\begin{array}{rcl} \Phi_\omega : \Theta_M & \longrightarrow & \mathcal{H}^{(-1)} \\ X & \longmapsto & \nabla_X P_\omega \end{array}$$

One might consider the covariant derivative of P_ω with respect to the vector field ∂_{t_1}. From the fact that the $\mathcal{H}^{(k)}$ define a filtration on the Gauß-Manin system it follows that $\nabla^k_{\partial_t} P_\omega \in \mathcal{H}^{(k-1)}$. The section $\nabla^k_{\partial_t} P_\omega$ defines a period map denoted by Φ^k_ω which is called k-th adjoint period map in [VG82].

Denote by $\phi_\omega^k := \Phi^k_{|M\setminus D}$ the restriction to a morphism from $\Theta_{|M\setminus D}$ to $\mathcal{H}$. A period map P_ω is called non-degenerate in [VG82] iff the morphism ϕ_ω is an isomorphism of vector bundles. It turns out that the non-degeneracy of a period map is determined by finite jets of the form ω and that under some hypothesis (see lemma 1.23 below), almost all forms give rise to non-degenerate period maps.

Suppose that we are given a form ω which yields an non-degenerate period map. Then we can use the bundle isomorphism $\Theta_{M\setminus D} \to \mathcal{H}$ to carry over existing structures in $\mathcal{H}$ onto the tangent bundle. Most important in the following is the *intersection form* on $\mathcal{H}$: this is a bilinear (possibly degenerate) pairing $I : \mathcal{H} \otimes \mathcal{H} \to \mathcal{O}_{M\setminus D}$ defined by the topological intersection form of n-cycles in the manifolds V_t. The pairing I is symmetric (resp. anti-symmetric) iff n is even (resp. odd). The following lemma, taken from [VG82] shows how I can be carried over to the tangent bundle of M.

Lemma 1.23. *Suppose that I is non-degenerate and anti-symmetric (the number of arguments of f is even). Then μ is even and we have*

- *Almost all forms ω yield non-degenerate period maps, i.e. forms with degenerate P_ω^k form in the jet space an analytic subset.*
- *Let $\omega \in \Omega^n_{M\times X}$ such that P_ω^k is non-degenerate. Then there is an anti-symmetric form induced on $\Theta_{M\setminus D}$. For $k = \frac{n+1}{2} - 1$, this form extends to a holomorphic form on Θ_M which is a closed differential form on M, i.e. a symplectic structure. We call it intersection form on M.*

Proof. For the proof of both parts of the theorem, one needs to study the behavior of integrals of the type $\int_{\gamma_j} \nabla_{\partial_{t_i}} P_\omega^k$ where $\gamma_1, \dots, \gamma_\mu$ is a basis of horizontal sections of the homology bundle. The period map P_ω^k is non-degenerate iff the determinant of the matrix $J := (\int_{\gamma_j} \nabla_{\partial_{t_i}} P_\omega^k)_{i,j}$ (this is the Jacobi matrix of the period map) does not vanish outside the discriminant. This determinant is not a single-valued function in $M\setminus D$, but its square is invariant under the monodromy. One can prove that $\det^2(J)$ depends on finite jets of the form ω and vanishes outside D only for a proper subset in the jet space.

For the second statement, it is clear that the intersection form induces a non-degenerate antisymmetric pairing on $\Theta_{M\backslash D}$. We first have to prove that it extends over the discriminant. It suffices to show that it extends over the smooth points of the discriminant because then an extension over the whole of D exists by Hartog's theorem. So let p_0 be in D_{reg}. Let l be a line through p_0 in the ∂_{t_0}-direction. Then for $p \in l$ near p_0, the manifold V_p is a bouquet of μ n-spheres. We can choose a basis of the cohomology of this manifolds, consisting of cycles $\gamma_1, \gamma_2, \ldots, \gamma_\mu$ where γ_1 is the unique cycle vanishing at p_0, and the intersection form is given by $I(\gamma_1, \gamma_2) = 1$ and $I(\gamma_i, \gamma_j) = 0$ for $i, j \in \{3, \ldots, \mu\}$. Obviously, these cycles can be extended to horizontal sections of the homology bundle over l. Then it is known that the integrals $\int_{\gamma_i} P_\omega$ can be expanded in a power series in $p - p_0$ of the form (see also lemma 1.22 on page 41)

$$\int_{\gamma_1} P_\omega = (p-p_0)^{\frac{n+1}{2}} \sum_{i=0}^{\infty} A_i (p-p_0)^i$$

$$\begin{aligned} \int_{\gamma_2} P_\omega &= \tfrac{1}{2\pi i} \log(p-p_0)(p-p_0)^{\frac{n+1}{2}} \sum_{i=0}^{\infty} B_i (p-p_0)^i \\ &\quad + \sum_{i=0}^{\infty} C_i (p-p_0)^i \end{aligned}$$

$$\int_{\gamma_j} P_\omega = \sum_{i=0}^{\infty} D_i (p-p_0)^i \quad \forall\, i \in \{3, \ldots, \mu\}$$

where A_i, B_i, C_i, D_i are locally constant sections of the cohomology bundle over l. If we consider the Jacobi matrix $\widetilde{J}$ of the k-th adjoint period map, then the intersection form on $M\backslash D$ is given by $\widetilde{J}^T I \widetilde{J}$ where I is the matrix of the intersection form in the cohomology bundle in a basis dual to γ_i. Therefore, for $k \leq \frac{n+1}{2} - 1$, $\widetilde{J}^T I \widetilde{J}$ can be extended over D_{reg} and hence over D. It remains to prove that it is closed and non-degenerate near the origin. We have to prove that $\det(\widetilde{J}^T I \widetilde{J})$ does not vanish, but this is clear since $\det(\widetilde{J}^T I \widetilde{J}) = \det^2(\widetilde{J}) \det(I)$, I is locally constant and the order of $\det^2(\widetilde{J})$ equals $\mu(n - 2k - 1)$ which is zero for $k = \frac{n+1}{2} - 1$. From the fact that the intersection form I is locally constant it follows that the induced form on M is closed. This finishes the proof. □

Definition 1.24. *Suppose that we are in the situation of the lemma, i.e., that we have a symplectic structure on M. Then let $\delta = \frac{\mu}{2}$ and denote* *by $B^\delta \subset D$ the closure of the set of points $t \in M$ such that $f^{-1}(t)$ has exactly δ A_1-singularities. We call this subspace the $\mu/2$-stratum.*

Note however that is is unclear whether this space is always non-empty. In the case of curves, it is itself a subspace of dimension $\mu/2$, thus non-empty (see the last remark of this section). Now we prove the main theorem of this section, which is also due to Givental and Varchenko ([VG82]).

Theorem 1.25. *The $\mu/2$-stratum is a lagrangian subvariety with respect to the symplectic structure of M.*

Proof. Let $p_0 \in B^\delta_{reg}$ and $U \subset B^\delta_{reg}$ an open neighborhood of p_0 in B^δ_{reg}. Identify $T_{p_0}M$ with M near p_0 and set $W := \{q_0 + s\partial_{t_0} \mid q_0 \in U\,;\ s \in [0,\epsilon) \subset \mathbb{R}_{\geq 0}\}$. For ϵ and U small enough, the intersection $(W\backslash U) \cap D$ where D is the discriminant will be empty. We proved in lemma 1.22 on page 41 that the k-th adjoint period map is a section of $\mathcal{H}^{(k-1)}$. By choosing a trivialization of this bundle over W, the period map P^k_ω can be written as a family of maps $P_t : U \to H := H^n(V_p, \mathbb{C})$ where V is a fixed Milnor fibre for $p = p_0 + s\partial_{t_0} \in W\backslash U$. In H we can chose a special basis: There are δ cycles vanishing at $p_0 \in U$. These cycles vanish at different points of $f^{-1}(p_0)$, so they do not intersect in H (for s sufficiently small). Denote them by $\gamma_1, \ldots, \gamma_\delta$. The intersection form I was supposed to be symplectic, so there are complementary cycles $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_\delta$ such that $I(\gamma_i, \widetilde{\gamma}_j) = \delta_{ij}$ (and $I(\gamma_i, \gamma_j) = 0$, $I(\widetilde{\gamma}_i, \widetilde{\gamma}_j) = 0$). Then we have

$$\int_{\gamma_i} P_\omega = (p-p_0)^{\frac{n+1}{2}} \sum_{i=0}^\infty A_i (p-p_0)^i = s^{\frac{n+1}{2}} \sum_{i=0}^\infty A_i s^i$$

$$\int_{\widetilde{\gamma}_i} P_\omega = \log(s) s^{\frac{n+1}{2}} \sum_{i=0}^\infty B_i s^i + \sum_{i=0}^\infty C_i s^i$$

In particular, we get that $\int_{\gamma_i} P^k_\omega$ (remember that $k = \frac{n+1}{2} - 1$) is zero on $U \times \{0\} \subset W$, that is, $P_0(U)$ is zero on the cycles γ_i. Therefore, also the intersection form I is zero on the image of $P_0(U)$. This implies that the form induced on W (recall that it was defined on the discriminant D by analytic continuation of the form on $M\backslash D$) vanishes on U. □

We make only two additional remarks on the singularities B^δ: First, in the case of the A_{2n}-singularity the spaces B^δ obviously coincide with the B^n's defined above. It remains to prove that the map R carries the intersection form to the form coming from the representation of $\mathfrak{sl}_2$. Givental proves this in an indirect way, in fact, he shows that the symplectic form on $\mathcal{P}_{2n+1}$ relative to which the "first" open swallowtail Σ_2 is lagrangian is unique up to a constant factor. As R carries B^2 to Σ_2, and B^2 and Σ_2 are lagrangian with respect to the two symplectic forms, it follows that R is a symplectomorphism.

The second remark concerns the case $n = 1$, then M is the semi-universal deformation space of a plane curve singularity $(C, 0)$, and B^δ is the subspace of points t such that the deformed curve C_t is the image of a deformation of the normalization $\widetilde{C}$ of the original curve. B^δ is called δ-constant stratum, and the number δ is the usual δ-invariant of the normalization $n : \widetilde{C} \to C$. The normalization of B^δ (which is smooth by work of Teissier [Tei77]) is the semi-universal deformation space of the map n. In particular, in this case the space B^δ is non-empty.

1.6 Further examples

In this last section we mention very briefly other classes of lagrangian singularities. Much more could be said on these examples, but a detailed description is beyond the scope of this thesis.

1.6.1 Spectral covers of Frobenius manifolds

Frobenius manifolds have become a very active field of research in the last years. Manifolds with multiplication on the tangent bundle and compatible flat metric have first been introduced by K. Saito around 1980 (a good survey of Saito's work is [Oda87]). The very definition of a Frobenius manifold is due to Dubrovin (see, e.g., [Dub96]). We give the definition of a Frobenius manifold and show how to associate to it in a canonical way a lagrangian subvariety of the cotangent bundle.

Definition 1.26. *Let M be a complex-analytic manifold and g a flat metric, i.e. a symmetric and non-degenerate $(2, 0)$-tensor such that the associated Levi-Civita connection ∇ is flat. Let a commutative and associative multiplication on the tangent bundle Θ_M (that is, a symmetric $(2, 1)$-tensor Ω) be given. We write $X \circ Y := \Omega_X(Y)$ for all $X, Y \in \Theta_M$.*

Suppose that we have a global unit field e. Let the following conditions be satisfied:

- *The metric is compatible with the multiplication, that is, $g(X \circ Y, Z) = g(X, Y \circ Z)$ for $X, Y, Z \in \Theta_M$.*
- $\nabla\Omega = 0$.
- *The unit field e is horizontal, i.e., $\nabla e = 0$.*

Then $(M, \circ, g, e)$ is called a Frobenius manifold. Suppose moreover that there is a field $\mathfrak{E}$ with $\mathrm{Lie}_{\mathfrak{E}}(\circ) = d \cdot \circ$ and $\mathrm{Lie}_{\mathfrak{E}}(g) = D \cdot g$ ($d, D \in \mathbb{C}$, $d \neq 0$) and such that the endomorphism $\nabla\mathfrak{E} : \Theta_M \to \Theta_M$ which sends a vector field $X \in \Theta_M$ to $\nabla_X \mathfrak{E}$ is horizontal. Then we call $(M, \circ, g, e, \mathfrak{E})$ a Frobenius manifold with conformal structure and $\mathfrak{E}$ its Euler field.

Consider the symmetric algebra $S^\bullet(\Theta_M)$ of Θ_M. This is a sheaf of algebras which can be canonically identified with the subsheaf of $\mathcal{O}_{T^*M}$ consisting of functions on the cotangent bundle which are polynomial with respect to the fibers of the projection $T^*M \to M$ (see lemma B.2 on page 175). The multiplication tensor can be seen as a morphism $\Theta_M \to \mathcal{E}nd(\Theta_M)$. It extends by composition to the tensor algebra $T^\bullet(\Theta_M)$ and descends due to commutativity to $S^\bullet(\Theta_M)$. The morphism

$$S^\bullet(\Theta_M) \longrightarrow \mathcal{E}nd(\Theta_M)$$

obtained in this way provides Θ_M with a $S^\bullet(\Theta_M)$-module structure. Therefore, the annihilator of Θ_M as a $S^\bullet(\Theta_M)$-module defines an ideal sheaf $\mathcal{I} \subset S^\bullet(\Theta_M)$. Denote its extension to $\mathcal{O}_{T^*M}$ also by $\mathcal{I}$.

Definition 1.27. *The subvariety $L \subset T^*M$ defined by $\mathcal{I} \subset \mathcal{O}_{T^*M}$ is called the spectral cover (or the analytic spectrum) of the Frobenius manifold M.*

One remarks that the analytic spectrum only depends on the multiplication but not on the metric. This fact is used extensively in the first part of [Her02], where manifolds $(M, \circ, e, \mathfrak{E})$ without metric are studied (they are called F-manifolds). The following theorem relates Frobenius manifolds with lagrangian subvarieties.

Theorem 1.28. *Let the multiplication* $\circ$ *be generically semi-simple, that is, suppose that generically one can finds local coordinates* $(q_1, \dots, q_n)$ *on* M *such that* $\partial_{q_i} \circ \partial_{q_j} = \delta_{ij}$. *Then the spectral cover* L *is a reduced subvariety of the cotangent bundle* T^*M *which is a lagrangian on its smooth locus.*

The proof can be found in [Aud98b] or [Aud98a]. Frobenius manifolds with generically semi-simple multiplication are also called *massive*.

There are essentially two main classes of examples of Frobenius manifolds: Quantum cohomology and unfolding of singularities. In the first case, the manifold M is the total cohomology in even degree $H^{2*}(X, \mathbb{C})$ (one can define it on the whole cohomology using super-structures) of a smooth projective manifold X (there is also a more general definition working for any symplectic manifold). The metric is simply the intersection form, which is obviously flat. However, the product comes from the so called genus zero Gromov-Witten invariants and is a multiplication of two elements $\alpha, \beta \in H^{2*}(X, \mathbb{C})$ depending on a third class $\xi \in H^{2*}(X, \mathbb{C})$. Therefore it defines a multiplication on the tangent bundle of M. However, it is not true that the Frobenius structure defined in this way is always massive, see [Aud98a] for a discussion of this fact. One might ask whether for manifolds with multiplication on the tangent bundle which is not semi-simple, the ideal defining the spectral cover (or even its radical) is still involutive.

For unfoldings of singularities, the situation is in some sense inverse to the one just described: The manifold M is the parameter space of a semi-universal unfolding (just like in section 1.5 on page 36) and the multiplication comes simply from the Kodaira-Spencer map of the unfolding. In fact, it is true in general that the spectral cover determines completely the multiplication. For a semi-universal unfolding, the spectral cover is isomorphic to the critical space of the family. Therefore it is a smooth space, and we are in the situation of the Arnold correspondence between lagrangian mappings and families of functions (see definition 1.6 on page 15). In particular, every germ of a Frobenius manifold with smooth analytic spectrum is a product of semi-universal unfoldings of hypersurface singularities.

However, the main difficulty to get a Frobenius structure on M in this case is the construction of the metric. One uses in principle the same theory as described in section 1.5, that is, a period map which identifies the tangent bundle of M with a certain locally free extension $\mathcal{H}^{(k)}$ of the cohomology bundle over $M \backslash D$ (D being the discriminant). Apart from the intersection form, there is a second topologically defined form in the fibres of the cohomology bundle, namely, the so called Seifert form. In contrast to the intersection form, it is always non-degenerate and symmetric. The main point now is to choose the right period map which transfers this form to the tangent bundle (it needs to define a *flat* metric on M). K. Saito's constructed such a map which comes form a section of $\mathcal{H}^{(k)}$ called the primitive form. Its construction is rather subtle and uses deep results from algebraic analysis. One can consult the original articles of K. Saito as well as [Oda87] or [Her02] for a more simplified treatment.

1.6.2 Special lagrangian singularities

Let us consider the complex linear space $\mathbb{C}^n$ as a real symplectic manifold (thus, as $\mathbb{R}^{2n}$) with symplectic form given by $\omega = \sum_{i=1}^n dz_i \wedge d\overline{z}_i$, where $z_1, \ldots, z_n$ are complex coordinates. Then we can speak about real lagrangian submanifolds (or subvarieties) of $\mathbb{C}^n$. On the other hand, the presence of a complex structure makes it possible to distinguish certain of these lagrangian submanifolds.

Definition 1.29. *A special lagrangian submanifold of $\mathbb{C}^n$ is a (real) n-dimensional submanifold L such that the symplectic form ω and the imaginary part of the holomorphic n-form $\Omega := dz_1 \wedge \ldots \wedge dz_n$ vanish on L.*

This definition comes from the so-called *calibrated geometry*, namely, special lagrangian submanifolds are characterized by the condition that they are area-minimizing, in the sense that they admit an orientation such that at each point $p \in L$, we have $Re(\Omega)_{|T_pL} = vol_{|T_pL}$, where vol is the natural volume form given by the metric on $\mathbb{C}^n$ and the orientation of L. This definition can be found in [HL82].

It should be noticed that in the above definition, the fact that L is a submanifold of $\mathbb{C}^n$ is not really used. The only point is the existence of a holomorphic n-form. This leads to the more general notion of a special lagrangian submanifold of a Calabi-Yau manifold.

Definition 1.30. *Let X be a Calabi-Yau manifold of dimension n, that is, a (complex) n-dimensional Kähler manifold which admits a non-vanishing holomorphic differential form Ω of degree n. Then $L \subset X$ is called special lagrangian iff it is lagrangian with respect to the Kähler form ω and iff $Im(\Omega)_L = 0$.*

For other characterizations of Calabi-Yau manifolds, see the discussion of applications of the T^1-lifting theorem in the first appendix, in particular corollary A.25 on page 157. In [Joy00], an even more general notion, that of an *almost Calabi-Yau manifold* is used.

We informally define singular special lagrangian subvarieties of Calabi-Yau manifolds as varieties whose smooth locus is a special lagrangian submanifold. The interest in these varieties comes from the so called SYZ-conjecture (after Strominger, Yau and Zaslow): It is expected that mirror symmetry can be expressed as a duality between two maps $f : M \to B$ and $f^* : M^* \to B$, where M and M^* is a (mirror) pair of Calabi-Yau 3-folds, B is a real three-dimensional manifold and the maps f and f^* are fibrations in special lagrangian three-tori over an open dense subset B_0. The main problem is to understand what happens over $B \backslash B_0$. It is unknown in general what type of degenerations can occur. The reader can consult [Joy00] and the reference therein for further details concerning singularities of special lagrangians. We restrict ourselves here to one simple example, which can already be found in [HL82].

Consider the following map

$$\begin{array}{rcl} f : \mathbb{C}^3 & \longrightarrow & \mathbb{R}^3 \\ (z_1, z_2, z_3) & \longmapsto & (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, Im(z_1 z_2 z_3)) \end{array}$$

The zero fibre of this map (denote it by L_0) can be described geometrically as follows: Consider a three-dimensional (real) torus T^3, given by the equations $|z_i|^2 = 1$ as lying in the five-dimensional sphere S^5 of radius $\sqrt{3}$. Then cut this torus with the subspace given by $z_1 z_2 z_3 = 1$. This yields a two- dimensional subtorus T^2 of T^3. Finally, take the cone over this manifold inside $\mathbb{R}^6$, that is, the set of all real lines through the origin and points of T^2. This cone is diffeomorphic to L_0.

It follows from the construction that L_0 has a unique singular point at the origin in $\mathbb{R}^6$. We chose coordinates (x, y, z, w, p, q) of $\mathbb{R}^6$ such that

$z_1 = x + iy$, $z_2 = z + iw$ and $z_3 = p + iq$, so that $\omega = dx \wedge dy + dz \wedge dw + dp \wedge dq$ and

$$\Omega = dx \wedge dz \wedge dp - (dx \wedge dw \wedge dq + dy \wedge dw \wedge dp + dy \wedge dz \wedge dq)$$

$$+i\,(dx \wedge dz \wedge dq + dx \wedge dw \wedge dp + dy \wedge dz \wedge dp - dy \wedge dw \wedge dq)$$

Then L_0 is given by the following equations:

$$f_1 := z^2 - p^2 + w^2 - q^2$$

$$f_2 := x^2 - p^2 + y^2 - q^2$$

$$f_3 := zpy + xpw + xzq - ywq$$

Its singular locus (reduced structure) is given by the vanishing of

$$\begin{array}{ccc} x^2 + y^2 & p^2 + q^2 & z^2 + w^2 \\ pw + zq & py + xq & zy + xw \\ zp - wq & xp - yq & xz - yw \end{array}$$

This shows that over $\mathbb{R}$, the singular locus is a point whereas over $\mathbb{C}$, it is of dimension two. In fact, a primary decomposition shows that it is a union of three components, given by the following ideals:

$$I_1 = (w, z, py + xq, p^2 + q^2, xp - yq, x^2 + y^2)$$

$$I_2 = (q, p, zy + xw, z^2 + w^2, xz - yw, x^2 + y^2)$$

$$I_3 = (y, x, pw + zq, p^2 + q^2, zp - wq, z^2 + w^2)$$

Chapter 2

Lagrangian deformations

In this chapter we start to investigate deformation theoretic questions for lagrangian singularities. To motivate constructions which will be introduced later, we first discuss two simple cases, namely, that of smooth real lagrangian submanifolds and that of plane curve singularities. Here the deformation theory is considerably simpler to describe than in the general case and more or less complete results are already known. In the third section of this chapter, we introduce a general framework covering all deformation problems associated to singular lagrangian subvarieties. We work in the context of categories fibred in groupoids and deformation functors, which we explain in some detail in Appendix A. The aim of the first two sections is to describe infinitesimal deformations in a "naive" sense, that is, we consider deformations over the double point up to an appropriate group action (which takes into account the symplectic structure). The more intrinsic meaning of these deformations spaces as tangent spaces of a functor will become clear in the general case discussed in the third section and in the next two chapters.

2.1 Real lagrangian submanifolds

We state and prove a classical result concerning deformations of real lagrangian submanifolds $L \subset (M, \omega)$. The setup is as follows: One starts with a symplectic C^∞-manifold (M, ω) (which we suppose to be simply connected for simplicity) and a (smooth) lagrangian submanifold $L \subset M$. Recall the following theorem (see, e.g., [Wei73]).

Theorem 2.1. *There is an open (tubular) neighborhood U of L in M, an open neighborhood V of the zero section $T^*_L L \subset T^* L$ and a symplectomorphism $\Phi : U \to V$ such that $\Phi(L) = T^*_L L$.*

We denote by $\mathfrak{L}(M, \omega)$ the space of all lagrangian submanifolds of M. This space can be equipped with a topology, see [Wei73]. Now we consider a deformation of L in M, that is, a map $\gamma : I \to \mathfrak{L}(M, \omega)$ where I is an interval in $\mathbb{R}$ containing zero such that $\gamma(0) = L$ and such that $\gamma(t) \subset U$ for all $t \in I$.

Using the symplectomorphism Φ, we get a one-parameter family β_t of sections of T^*L, that is, a family of differential forms on L. Moreover, any $\gamma_t \in \mathfrak{L}(M, \omega)$ is lagrangian, meaning that $\beta_t^* \omega = 0$, but $\beta_t^* \omega = \beta_t^*(d\alpha) = d\beta_t$ where α is the Liouville form on $V \subset T^*L$. Therefore, we obtain a family of closed forms on L. Suppose that β_t are *exact* one forms, i.e., that there is a family of functions $F_t : L \to \mathbb{R}$ with $dF_t = \beta_t$. Then the flow of the (time-dependent) hamiltonian field X_{F_t} defines a family of symplectomorphisms of V, thus, a family of symplectomorphisms of U trivializing the family γ_t. This shows that the space of deformations of L coincides with the space of maps $\widetilde{\Phi}$ from I to $H^1(L, \mathbb{R})$, the first de Rham cohomology group of L. In particular, infinitesimal deformations are given by vectors $\frac{\partial}{\partial t} \widetilde{\Phi}_{t=0}$. Therefore we have

Theorem 2.2. *The infinitesimal deformation space of a smooth lagrangian submanifold $L \subset M$ is naturally isomorphic to $H^1(L, \mathbb{R})$.*

2.2 Curve singularities

In this section, we will discuss another simple example of lagrangian deformations where deformation spaces can be calculated "by hand": germs of curves in the plane (seen as a symplectic manifold by any volume form $\omega \in \Omega_{\mathbb{K}^2,0}$). Such a curve is obviously a lagrangian subspace and moreover, any deformed curve is still lagrangian. However, the automorphism group acting is the symplectic group which is strictly smaller than the usual automorphism group (the one used for V-equivalence, also called contact equivalence). Therefore, it is natural to expect the space of (infinitesimal) deformations of a lagrangian curve singularity to be *bigger* then the usual T^1. This is indeed the case and can be seen as follows: Denote the singularity by $(C,0) \subset (\mathbb{K}^2,0)$ and suppose it is given by $f \in \mathcal{O}_{\mathbb{K}^2,0}$. Then any deformation over $\mathbb{K}[\epsilon]/\epsilon^2$ is given by a equation of type $f + \epsilon\widetilde{f}$ with $\widetilde{f} \in \mathcal{O}_{\mathbb{K}^2,0}$. But any $\widetilde{f} \in (f)$ is a trivial deformation because then the ideals (f) and $(f+\epsilon\widetilde{f})$ are the same in $\mathcal{O}_{\mathbb{K}^2,0}[\epsilon]/\epsilon^2$. So deformations are parameterized by $\mathcal{O}_{C,0}$. But some of them are still trivial, namely, those induced from hamiltonian vector fields in $\mathbb{K}^2$. The space of these fields is again parameterized by $\mathcal{O}_{C,0}$ since elements from (f) give hamiltonian fields tangent to C. We see that the space of infinitesimal lagrangian deformations of the curve germ $(C,0)$, which we denote by $T^1_{LagDef}(C,0)$ is given as the cokernel of the map

$$\begin{array}{rcl} \delta : \mathcal{O}_{C,0} & \longrightarrow & \mathcal{O}_{C,0} \\ h & \longmapsto & \{h,f\} \end{array}$$

As we have a canonically given non-degenerate two-form ω, there is an isomorphism $\mathcal{O}_{\mathbb{K}^2,0} \cong \Omega^2_{\mathbb{K}^2,0}$ which does not depend on any choice. The Poisson bracket on $\mathcal{O}_{\mathbb{K}^2,0}$ of two elements h_1, h_2 corresponds under this isomorphism to $dh_1 \wedge dh_2$. Therefore, we get

$$T^1_{LagDef}(C,0) \cong \frac{\Omega^2_{\mathbb{K}^2,0}}{f\Omega^2_{\mathbb{K}^2,0} + df \wedge d\mathcal{O}_{\mathbb{K}^2,0}} \tag{2.1}$$

This last quotient is a quite familiar object in singularity theory, if we see f as a mapping germ $f : (\mathbb{K}^2,0) \to (\mathbb{K},0)$, then $\Omega^2_{\mathbb{K}^2,0}/df \wedge d\mathcal{O}_{\mathbb{K}^2,0}$ is the germ of sections of a free $\mathcal{O}_{\mathbb{K},0}$-module, called the Brieskorn lattice of f and denoted by $''H$. The rank of this module equals μ, the Milnor number of the singularity f. Thus we obtain

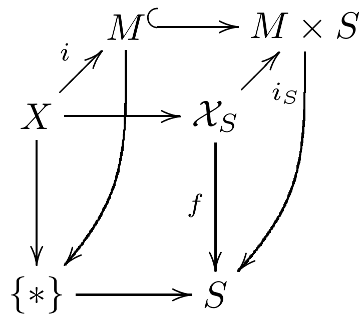
Theorem 2.3. *The space of infinitesimal lagrangian deformations of a germ of a plane curve $(C,0)$ given by an equation $f \in \mathcal{O}_{\mathbb{K}^2,0}$ up to symplectomorphisms of $\mathbb{K}^2$ is canonically isomorphic to the zero fibre of the Brieskorn lattice of f. In particular, $T^1_{LagDef}(C,0)$ is a μ-dimensional vector space.*

This result is remarkable in several ways: First, the usual infinitesimal deformation space $T^1_{Def}(C,0)$ is of dimension τ, the Tjurina number of $(C,0)$. Recall that $\tau = \dim_{\mathbb{K}} \mathcal{O}_{\mathbb{K}^2,0}/\left(f, \partial_{x_1} f, \ldots, \partial_{x_n} f\right)$. We have the equality $\tau = \mu$ iff f is quasi-homogeneous with positive weights (see [Sai71]). In general, the Milnor number is greater than the Tjurina number. In that case we see that the space of symplectic structures modulo symplectomorphisms which leave the curve C invariant is of dimension $\mu - \tau$. This also follows from results of Givental (see [Giv88]), in fact, he proves that (in the complex case), there is at most one symplectic structure ω (up to symplectomorphisms fixing C) for a given class $[\alpha] \in H^1(\widetilde{\Omega}^\bullet_C)$ such that $d\alpha = \omega$ and $\omega_{C_{reg}} = 0$.

2.3 The lagrangian deformation functor

Motivated by the two above examples, we will now define a very general framework which covers different deformation problems associated to lagrangian subvarieties. More precisely, consider a mapping (which might not be an embedding) $i : X \to M$ of a (not necessarily smooth) reduced analytic space X into a symplectic manifold (M, ω) over $\mathbb{K}$, such that the image is lagrangian, that is, such that $i^*\omega = 0$ where we see $i^*\omega$ as an element of $\widetilde{\Omega}^2_X$, the module of Kähler two-forms on X modulo torsion (see also the discussion on page 20). Denote by **Art** the category of Artin rings.

Definition 2.4. *Let a mapping $i : X \to M$ as above be given. We define a category co-fibred in groupoids over* **Art** *(denoted by* $\mathbf{LagIso}_{X/M}$*) as follows: Its objects are diagrams of the following type*



with $S \in \mathbf{Art}^{opp}$*,* $f : \mathcal{X}_S \to S$ *flat and* $\left(pr \circ \widetilde{i}\right)^* \omega = 0 \in \widetilde{\Omega}^2_{\mathcal{X}_S/S}$, *where* $pr : M \times S \to M$ *is the projection. Morphisms (over a morphism* $S' \to S$ *in* Art^{opp}*) are the obvious (huge) commutative diagrams connecting two of the above diagrams, where the map* $M \times S' \to M \times S$ *is fibrewise symplectic and induces the identity over* $\{*\}$*. It is easily checked that* $\mathbf{LagIso}_{X/M}$ *is indeed a category co-fibred in groupoids. As explained in Appendix A (see section A.1.2 on page 138), we get a functor* $LagIso_{X/M} \in Fun$ *by associating to* $S \in \mathbf{Art}^{opp}$ *the set of isomorphism classes of elements of* $\mathbf{LagIso}_{X/M}(S)$*.*

The name **LagIso** is chosen according to the two particular deformation problems covered by this functor: deformations of lagrangian subvarieties and of isotropic mappings, see definition 2.6 on the next page.

In order to fit into the general pattern as described in Appendix A, we need to check some technical properties of the functor $LagIso_{X/M}$.

Lemma 2.5. $LagIso_{X/M}$ *satisfies the axioms (H1) and (H2) from definition A.7 on page 141 and also axiom (H5) from definition A.19 on page 152.*

Proof. (H5) obviously implies (H1). Moreover, once we have proved (H5), the bijectivity in (H2) follows immediately as in the prove for the case of flat deformations in [Art76]. We use the prove of (H5) in [Gro97]. So let us be given surjections $A' \to A$ and $A'' \to A$ in **Art** and deformations $(X_{A'}, i_{A'}) \in LagIso_{X/M}(A')$, $(X_{A''}, i_{A''}) \in LagIso_{X/M}(A'')$ and $(X_A, i_A) \in LagIso_{X/M}(A)$ with

$$\mathcal{O}_{X_{A'}} \otimes_{A'} A = \mathcal{O}_{X_{A''}} \otimes_{A''} A = \mathcal{O}_{X_A}$$

and such that the $\mathcal{O}_{M\times R}$-module structure of $\mathcal{O}_{X_R}$ $(R = A, A', A'')$ representing the morphism i_R is compatible with these tensor products. Then we set

$$\tau_{A',A'',A}\left(X_{A'}, X_{A''}\right) := \widetilde{X} := \left(X, \mathcal{O}_{X_{A'}} \times_{\mathcal{O}_{X_A}} \mathcal{O}_{X_{A''}}\right)$$

We see that there is a natural algebra morphism $\widetilde{i}^* : \mathcal{O}_{M\times \widetilde{A}} \to \mathcal{O}_{\widetilde{X}}$, where $\widetilde{A} := A'' \times_A A'$ (because $\mathcal{O}_{M\times \widetilde{A}} \cong \mathcal{O}_{M\times A''} \times_{\mathcal{O}_{M\times A}} \mathcal{O}_{M\times A'}$). Then obviously $\widetilde{i}^* \circ (pr)^*\omega = 0$, because this pullback is a relative form which is zero on the factors of the fibred sum. $\square$

The following two chapters are devoted to study special cases of the functor *LagIso*. For notational convenience, we define functors which distinguish these two cases.

Definition 2.6. *Let* (M, ω) *a* $2n$*-dimensional symplectic manifold over* $\mathbb{K}$*.*

- *Let* $L \subset M$ *be a lagrangian subvariety, given by an involutive ideal sheaf* $\mathcal{I} \subset \mathcal{O}_M$*. Then we denote by* $LagDef_L$ *the functor* $LagIso_{L/M}$ *associated to the embedding* $i : L \hookrightarrow M$*. The elements of* $LagDef_L(S)$ *for* $S \in \mathbf{Art}^{opp}$ *are isomorphism classes of deformed ideals* $\widetilde{\mathcal{I}} \subset \mathcal{O}_{M \times S}$ *which are involutive with respect to the Poisson-structure on* $M \times S$ *up to the action of relative symplectomorphisms in* $M \times S$*.*
- *Consider an isotropic mapping* $i : X \to M$ *(i.e.* $i^*\omega = 0$*) where* X *is an open subset of* $\mathbb{K}^n$*. Then we let* $IsoDef_i := LagIso_{X/M}$ *be the functor of deformations of the mapping* i*.* $IsoDef_i(S)$ *are deformed isotropic maps* $\widetilde{i} : X \times S \to M \times S$ *(i.e.* $\widetilde{i}^* \circ pr^*\omega = 0$*) up to the action of the group which is the semi-direct product of the group* $Aut_S(X \times S)$ *with* $Symp_S(M \times S)$*.*

It should be clear that the functor *LagIso* reduces in the two particular cases to the functors *LagDef* and *IsoDef*: in the first case, if $i : L \hookrightarrow M$ is an embedding, then by flatness a deformation $i_S : L_S \to M \times S$ of this map will still be an embedding, that is, $LagIso_{L/M}$ consists of deformations of the subspace L inside M. On the other hand, if X is open in $\mathbb{K}^n$, it does not deform at all, so elements of $LagIso_{X/M}$ are isomorphism classes of mappings $i : X \times S \to M \times S$.

We remark that one can of course define local versions of these functors, that is one starts with germs of objects of the above type. This is indeed the case that we will consider mainly in the next two chapters. However, we can always work with the functors as defined by supposing that L, M and the mapping i are small representatives for the given germs (L and M have to be Stein in the complex case).

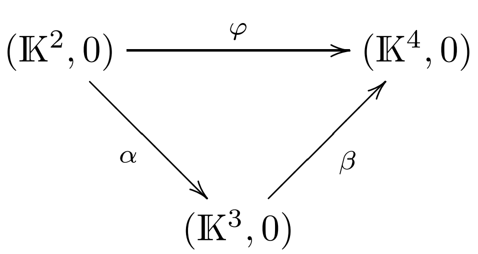
When defining the functor $LagIso_{X/M}$ for a general mapping $i : X \to M$, one may ask whether there are such maps where X is not smooth and i is not an embedding. In the following theorem, we give an example.

Theorem 2.7. *Fix a positive integer n and let $(X,0) \subset (\mathbb{K}^3,0)$ be the three-dimensional A_n-singularity, given by the equation $xz - y^{n+1} = 0$. Then the map germ*

$$\begin{array}{rcl} \beta : (\mathbb{K}^3,0) & \longrightarrow & (\mathbb{K}^4,0) \\ (x,y,z) & \longmapsto & (x,y,zy,xy) \end{array}$$

defines an isotropic map $(X,0) \to (\mathbb{K}^4,0)$, i.e., we have $(\beta^*\omega)_{|X_{reg}} = 0$.

Proof. Consider the following commutative diagram of map germs



where

$$\begin{array}{rcl} \alpha : (\mathbb{K}^2,0) & \longrightarrow & (\mathbb{K}^3,0) \\ (s,t) & \longmapsto & (s^{n+1}, st, t^{n+1}) =: (x,y,z) \end{array}$$

is the normalization of $(X,0)$ and

$$\begin{array}{rcl} \varphi : (\mathbb{K}^2,0) & \longrightarrow & (\mathbb{K}^4,0) \\ (s,t) & \longmapsto & (s^{n+1}, t^{n+1}, st^{n+2}, s^{n+2}t) \end{array}$$

is the composition. Then $\varphi^*\omega = 0$. This proves the theorem. □

There is one deformation problem we are going to consider which is not covered by the above formalism, namely, deformations of an integrable system. In principle this problem can also be seen as a special version of the functor *LagIso* by using the graph construction, but this needs supplementary effort to be written down properly, without being very useful in applications. Therefore, we will define an extra functor, adapted for this problem. The relation with the deformation of lagrangian submanifolds via the graph construction will become clear later (see lemma 3.27 on page 81).

The definition of the deformation functor for an integrable system is rather simple: Let us consider a mapping

$$F = (f_1, \ldots, f_n) : M \longrightarrow U \subset \mathbb{K}^n$$

such that $\{f_i, f_j\} = 0$ for all i, j. We call a map

$$\widetilde{F} = (\widetilde{f}_1, \ldots, \widetilde{f}_n) : M \times S \longrightarrow U$$

with $\widetilde{F}(\mathbf{p}, \mathbf{q}, 0) = F(\mathbf{p}, \mathbf{q})$ an unfolding of F over S. We have a natural group action on the set of all unfoldings of F over S, namely, let Symp_S^{2n} be the group of all S-symplectomorphisms of $M \times S$ (i.e., the group of all families of symplectomorphisms of M, parameterized by S). This defines a groupoid $\mathbf{HamDef}_F(S)$ and one sees that $\mathbf{HamDef}_F$ becomes a category co-fibred in groupoids. Therefore, we obtain a functor HamDef_F by sending $S \in \mathbf{Art}^{opp}$ to $\mathrm{Iso}\,(\mathbf{HamDef}_F(S))$.

As the spaces involved here are smooth and therefore deform trivially, it is easy to check the following fact.

Lemma 2.8. *The functor HamDef_F satisfies conditions (H1), (H2) and (H5).*

We will see that this deformation functor is much simpler to handle than the functor *LagDef*. However, it is only of theoretical interest because its tangent space is almost never finite-dimensional.

Chapter 3

Lagrangian subvarieties

The first special case of the general lagrangian deformation problem described in the last chapter is concerned with lagrangian subvarieties L embedded in a symplectic manifold M. It turns out that the deformation theory of L is related to a "symplectic analogue" of the de Rham complex, namely, a sheaf complex on L which coincides with the de Rham complex on the smooth locus of L. This construction is a special case of the general formalism of Lie algebroids, which we introduce in the first section.

3.1 Lie algebroids

We give the definition of a Lie algebroid. We treat directly the relative case, i.e. Lie algebroids over morphisms $X \to S$ of complex spaces. Studying deformations of lagrangian families turns out to be quite useful (like in any deformation theory), and the relative version of the lagrangian de Rham complex can be directly deduced from Lie algebroids in the relative setting. This complex, defined for modules over arbitrary Lie algebroids is an analogue of the de Rham complex in (ordinary) $\mathcal{D}$-module theory (see Appendix B), namely, for a Lie algebroid $\mathfrak{g}$ one constructs a non-commutative algebra $\mathcal{D}_{\mathfrak{g}}$ of generalized differential operators and defines $DR(\mathcal{M})$ as $\mathcal{RH}om_{\mathcal{D}_{\mathfrak{g}}}(\mathcal{O}_X, \mathcal{M})$ for any module $\mathcal{M}$

over $\mathfrak{g}$.

3.1.1 Lie algebroids and differential operators

First we define Lie algebroids and generalized differential operators independently from each other. Both of them form categories in a natural way. We show that there is a pair of adjoint functors between these categories. The material of this section is essentially taken from [Käl98], [BB93] and [Rin63].

Definition 3.1. *Let S be an analytic space over $\mathbb{K}$, $X \to S$ a morphism of analytic spaces and $\mathfrak{g}$ a sheaf of $\mathcal{O}_S$-Lie algebras, that is, a sheaf of $\mathcal{O}_S$-algebras satisfying the usual relations for Lie algebras. Suppose moreover that $\mathfrak{g}$ is a coherent sheaf of $\mathcal{O}_X$-modules together with a fixed morphism of $\mathcal{O}_S$-Lie algebras (the structure morphism, also called anchor by various authors)*

$$\alpha : \mathfrak{g} \longrightarrow \Theta_{X/S} = \mathcal{D}er_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$$

such that for all $\delta_1, \delta_2 \in \mathfrak{g}$ and $f \in \mathcal{O}_X$ we have

$$[\delta_1, f\delta_2] = \alpha(\delta_1)(f)\delta_2 + f[\delta_1, \delta_2]$$

*Then we call $\mathfrak{g}$ a **Lie algebroid** relative to the morphism $X \to S$ (or Lie algebroid over X/S for short). Lie algebroids over X/S form a category: a homomorphism of $\mathcal{O}_X$-modules and $\mathcal{O}_S$-Lie algebras is a morphism of Lie algebroids iff it commutes with the structure morphisms.*

As usual, most interesting from the geometric viewpoint is the case $\mathcal{O}_S = \mathbb{K}$, then we have a Lie algebroid on X. The basic Lie algebroid is the (relative) tangent sheaf itself. For a smooth variety X, the tangent sheaf Θ_X and the structure sheaf $\mathcal{O}_X$ generate a non-commutative algebra, the ring of differential operators $\mathcal{D}_X$ (see Appendix B for some aspects of $\mathcal{D}$-module theory, in particular lemma B.1 on page 174). In the following, we define differential operators associated to any Lie algebroid.

Definition 3.2. *Let X be an analytic space over S. Then a ring of differential operators on X/S is a (non-commutative) $\mathcal{O}_S$-algebra $\mathcal{D}$ together with a filtration*

$$0 \subset \mathcal{D}(0) \subset \mathcal{D}(1) \subset \ldots \subset \mathcal{D}$$

such that $\mathcal{D}(m)\mathcal{D}(n) \subset \mathcal{D}(m+n)$, $\cup_{i=0}^{\infty}\mathcal{D}(i) = \mathcal{D}$ *and such that the associated graded ring*

$$gr(\mathcal{D}) := \oplus_{i=0}^{\infty}\mathcal{D}(i)/\mathcal{D}(i-1)$$

is a commutative $\mathcal{O}_S$*-algebra. Moreover, we require that there is an inclusion* $i : \mathcal{O}_X \to \mathcal{D}(0)$ *such that*

$$[i(\mathcal{O}_X), \mathcal{D}(n)] \subset \mathcal{D}(n-1)$$

One can define the category of differential operators on X/S where morphism are algebra homomorphisms respecting the filtrations. Then to any ring $\mathcal{D}$ we associate a Lie algebroid $\mathfrak{g}$ by setting

$$\mathfrak{g} := \{\delta \in \mathcal{D}(1) \mid [\delta, i(\mathcal{O}_X)] \subset i(\mathcal{O}_X)\}$$

Here the Lie bracket is the usual commutator of elements on $\mathcal{D}$ (one has to check that $\mathfrak{g}$ is stable under this commutator). The structure morphism $\alpha : \mathfrak{g} \to \Theta_{X/S}$ is defined as $\alpha(\delta)(f) := i^{-1}([\delta, f])$ for $\delta \in \mathfrak{g}, f \in \mathcal{O}_X$. That this defines in fact a derivation will be proved in a more general context below (see theorem 3.6 on page 63). We get a functor L from the category of differential operators to the category of Lie algebroids. The following construction gives a left adjoint to L.

Let a Lie algebroid $\mathfrak{g}$ be given. Define the $\mathcal{O}_X$-module $\widetilde{\mathfrak{g}} := \mathcal{O}_X \oplus \mathfrak{g}$, which becomes a Lie algebra under the following bracket

$$\begin{array}{rcccc} [\,,\,] & : & \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} & \to & \widetilde{\mathfrak{g}} \\ & & (h_1, g_1), (h_2, g_2) & \longmapsto & (\alpha(g_1)(h_2) - \alpha(g_2)(h_1), [g_1, g_2]) \end{array}$$

Consider the universal enveloping algebra $\mathfrak{U}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$ of $\widetilde{\mathfrak{g}}$ over $\mathcal{O}_S$, i.e. the quotient of the tensor algebra $T^{\bullet}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$ by the ideal generated by $\widetilde{x} \otimes \widetilde{y} - \widetilde{y} \otimes \widetilde{x} - [\widetilde{x}, \widetilde{y}]$ for $\widetilde{x}, \widetilde{y} \in \widetilde{\mathfrak{g}}$. Inside $\mathfrak{U}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$ we have the subalgebra of "elements from $\mathfrak{g}$ and $\mathcal{O}_X$", that is, the subalgebra generated by the image of $\widetilde{\mathfrak{g}}$ in $\mathfrak{U}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$. Denote this subalgebra by $\mathfrak{U}^{+}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$. Finally, we have to take into account the $\mathcal{O}_X$-module structure of $\mathfrak{g}$. Therefore we define $\mathcal{D}_{\mathfrak{g}}$ to be the quotient of $\mathfrak{U}^{+}_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$ by the ideal generated by elements of the form $h \otimes \widetilde{x} - h\widetilde{x}$, where $h \in \mathcal{O}_X$ and $\widetilde{x} \in \widetilde{\mathfrak{g}}$. The ring $\mathcal{D}_{\mathfrak{g}}$ is canonically filtered: We define a grading on the Lie algebra $\widetilde{\mathfrak{g}}$ by setting $deg(g) = 1$ and and $deg(h) = 0$ for $g \in \mathfrak{g}, h \in \mathcal{O}_X$. This induces a filtration by order on $T^\bullet_{\mathcal{O}_S}(\widetilde{\mathfrak{g}})$ and thus on $\mathcal{D}_\mathfrak{g}$. We denote the associated graded ring by $gr(\mathcal{D}_\mathfrak{g})$. It can be checked that this ring is commutative.

Lemma 3.3. *The functor D from Lie algebroids to differential operators defined in this way is left adjoint to L.*

Proof. Let $\mathfrak{g}$ be a Lie algebroid and $\mathcal{D}$ any ring of differential operators on X/S. Then any morphism

$$\Phi : \mathfrak{g} \longrightarrow L(\mathcal{D})$$

of Lie algebroids extends first uniquely to a morphism of Lie algebras $\widetilde{\Phi} : \widetilde{\mathfrak{g}} \to \widetilde{L(\mathcal{D})}$ and then to a $\mathcal{O}_S$-algebra homomorphism

$$\begin{array}{rcrcl} \widehat{\Phi} & : & T^\bullet_{\mathcal{O}_S}(\mathcal{O}_X \oplus \mathfrak{g}) & \longrightarrow & \mathcal{D} \\ & & (h_1, g_1) \otimes (h_2, g_2) & \longmapsto & h_1 h_2 + h_1 \Phi(g_2) + h_2 \Phi(g_1) \\ & & & & +\alpha(g_1)(h_2) + \Phi(g_1)\Phi(g_2) \end{array}$$

where $g_i \in \mathfrak{g}$ and $h_i \in \mathcal{O}_X$. Then it is easy to see that $\widehat{\Phi}$ vanishes on elements of the form

$$\begin{array}{c} (h_1, g_1) \otimes (h_2, g_2) - (h_2, g_2) \otimes (h_1, g_1) - \\ (\alpha(g_1)(h_1) - \alpha(g_2)(h_1), [g_1, g_2]) \quad \text{and} \end{array}$$

$$h \otimes (h_1, g_1) - (hh_1, hg_1)$$

and thus defines a unique algebra morphism $\mathcal{D}_\mathfrak{g} \to \mathcal{D}$. □

We proceed to imitate some of the known constructions and objects for ordinary differential operators. Denote by $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g})$ the symmetric algebra over $\mathcal{O}_X$ of $\mathfrak{g}$. On this algebra we have a *Poisson*-bracket, defined by the bracket on $\mathfrak{g}$ and the Leibniz rule. More precisely, denote by j the embedding $\mathfrak{g} \hookrightarrow \mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g})$ and define

$$\begin{array}{rcl} \{j(x), j(y)\} & = & j([x, y]) \\ \{f, g\} & = & 0 \\ \{j(x), g\} & = & \alpha(x) g \end{array}$$

for all $x, y \in \mathfrak{g}$ and $f, g \in \mathcal{O}_X$. Remark that $j(\mathfrak{g})$ generates $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g})$ as an algebra over $\mathcal{O}_X$, therefore the bracket is well defined by the above definitions and the Leibniz rule.

On the other hand, the general theory of filtered rings (see [Gab81] and [Bjö93]) shows that the graded ring $gr(\mathcal{D}_{\mathfrak{g}})$ also carries a natural Poisson bracket (which is defined essentially in the same way). Then the morphism

$$\mathfrak{g} \longrightarrow gr^1(\mathcal{D}_{\mathfrak{g}}) \hookrightarrow \mathcal{D}_{\mathfrak{g}}$$

extends to a morphism of Lie algebras (Poisson algebras) $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g}) \longrightarrow gr(\mathcal{D}_{\mathfrak{g}})$ which is surjective by construction. The following lemma (which is in fact a generalization of the Poincaré-Birkhoff-Witt theorem) is proved in [Rin63].

Lemma 3.4. *Let $\mathfrak{g}$ be locally free over $\mathcal{O}_X$. Then the natural morphism $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g}) \longrightarrow gr(\mathcal{D}_{\mathfrak{g}})$ is an isomorphism.*

A basic question concerns the coherence of $\mathcal{D}_{\mathfrak{g}}$, $gr(\mathcal{D}_{\mathfrak{g}})$ and $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g})$. The methods to prove coherence are the same as for ordinary differential operators, an indication of this fact is found in [Käl98].

Lemma 3.5. *$\mathcal{D}_{\mathfrak{g}}$, $gr(\mathcal{D}_{\mathfrak{g}})$ and $\mathcal{S}^\bullet_{\mathcal{O}_X}(\mathfrak{g})$ are coherent sheaves of rings.*

3.1.2 Modules over Lie algebroids

A module over a Lie algebroid is intuitively an $\mathcal{O}_X$-module $\mathcal{M}$ with an action of $\mathfrak{g}$ on $\mathcal{M}$, i.e., a bracket $[\ ,\ ] : \mathfrak{g} \times \mathcal{M} \to \mathcal{M}$ such that $[g, fm] = f[g,m] + \alpha(g)(f)m$ and $[fg, m] = f[g,m]$ for all $g \in \mathfrak{g}, f \in \mathcal{O}_X$ and $m \in \mathcal{M}$. This can be reformulated in the following way.

Theorem+Definition 3.6. *Consider a faithful $\mathcal{O}_X$-module $\mathcal{M}$, that is, suppose that the natural morphism*

$$\begin{array}{rcl} i : \mathcal{O}_X & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}) \\ h & \longmapsto & (m \mapsto h \cdot m) \end{array}$$

is injective.

- *The **linear Lie algebroid** associated to $\mathcal{M}$ is defined as follows: Denote by $\mathcal{D}(\mathcal{M})(1)$ the subsheaf of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{M})$ of all operators δ such that $[\delta, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})] \subset \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$. Then set*

$$\mathfrak{c}_X(\mathcal{M}) := \{\delta \in \mathcal{D}(\mathcal{M})(1) \mid [\delta, i(\mathcal{O}_X)] \subset i(\mathcal{O}_X)\}$$

The Lie bracket on $\mathfrak{c}_X(\mathcal{M})$ is just the commutator (well defined due to the Jacobi identity), whereas the structure morphism is $\alpha(\delta) := (f \mapsto i^{-1}([\delta, f]))$. Then $(\mathfrak{c}_X(\mathcal{M}), [\ ,\ ], \alpha)$ is a Lie algebroid.

- *Let $\mathfrak{g}$ be a Lie algebroid and $\mathcal{M}$ a (faithful) $\mathcal{O}_X$-module. Then a structure of a left $\mathfrak{g}$-module on $\mathcal{M}$ is by definition a morphism of Lie algebroids $\mathfrak{g} \to \mathfrak{c}_X(\mathcal{M})$.*

Proof. We have to show that the structure morphism is well defined, i.e. that $\alpha(\delta)$ is really an $\mathcal{O}_S$-derivation of $\mathcal{O}_X$. Let $f_1, f_2 \in \mathcal{O}_X$ and denote by ϕ_1, ϕ_2 the multiplication with f_1, f_2, respectively. Moreover, let $h_k := \alpha(f_k) = i^{-1}([\delta, f_k])$ $(k = 1, 2)$. Then, as $[\delta, i(\mathcal{O}_X)] \subset i(\mathcal{O}_X)$, we have

$$[\delta, \phi_1]\phi_2 = \phi_2[\delta, \phi_1]$$

that is

$$\delta(f_1 f_2 m) - f_2\delta(f_1 m) - f_1\delta(f_2 m) + f_1 f_2 \delta(m) = 0$$

for all $m \in \mathcal{M}$. Moreover

$$\begin{aligned} f_2\delta(f_1 m) - f_2 f_1 \delta(m) &= f_2 h_1 m \\ f_1\delta(f_2 m) - f_1 f_2 \delta(m) &= f_1 h_2 m \end{aligned}$$

These three equations give

$$\delta(f_1 f_2 m) - f_1 f_2 \delta(m) = [\delta, f_1 f_2]m = h_1 f_2 m + f_1 h_2 m$$

This proves $\alpha(f_1 f_2) = \alpha(f_1) f_2 + f_1 \alpha(f_2)$. On the other hand, for any $\delta \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ we have $\delta(sm) = s\delta(m)$ for $s \in \mathcal{O}_S$ and $m \in \mathcal{M}$, therefore $\alpha(\delta)(s) = 0$. So we get $\alpha(\delta) \in \mathcal{D}er_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$. ☐

Remark that a left $\mathfrak{g}$-module as defined is nothing else than a left module over $\mathcal{D}_{\mathfrak{g}}$. There is also a corresponding definition of a right $\mathfrak{g}$-module, but we will not give it here. The structure sheaf $\mathcal{O}_X$ is always a (left) module over the Lie algebroid $\mathfrak{g}$, because $\mathfrak{c}_X(\mathcal{O}_X) = \Theta_X$ and the structure morphism $\alpha : \mathfrak{g} \to \Theta_X$ is a morphism of Lie algebroids.

Very much like for ordinary differential modules, one defines coherent left $\mathfrak{g}$-modules to be those which are coherent over $\mathcal{D}_{\mathfrak{g}}$. This condition turns out to be equivalent to the local existence of good filtrations.

Therefore one can define the graded module $gr(\mathcal{M})$ of a coherent $\mathfrak{g}$-module $\mathcal{M}$. It is a $gr(\mathcal{D}_{\mathfrak{g}})$-module (in particular, it is a module over $S^\bullet(\mathfrak{g})$). The radical of the annihilator of $gr(\mathcal{M})$ is independent of the good filtration chosen. Suppose in the following that $\mathfrak{g}$ is a locally free $\mathcal{O}_X$-module. Then there is a linear space over X, called $\mathbf{Spec}(S^\bullet(\mathfrak{g}))$ (it is the spectrum of the algebra $S^\bullet(\mathfrak{g})$ in the algebraic case) and a projection $p : \mathbf{Spec}(S^\bullet(\mathfrak{g})) \to X$ such that $p_*\mathcal{O}_{\mathbf{Spec}(S^\bullet(\mathfrak{g}))} = S^\bullet(\mathfrak{g})$. The space $\mathbf{Spec}(S^\bullet(\mathfrak{g}))$ replaces the cotangent bundle in usual $\mathcal{D}$-module theory where X is smooth, in the sense that we have a Poisson bracket on $S^\bullet(\mathfrak{g})$ and that the following holds.

Lemma 3.7. *Denote by $\mathcal{J}(\mathcal{M}) \subset S^\bullet(\mathfrak{g})$ the radical of the annihilator of the $S^\bullet(\mathfrak{g})$-module $gr(\mathcal{M})$. Then $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$. The subvariety defined by $\mathcal{J}(\mathcal{M})$ is called the singular support or the characteristic variety of the coherent $\mathfrak{g}$-module $\mathcal{M}$.*

The proof follows from Gabbers theorem (see [Gab81]). We remark that in contrast to the case $\mathfrak{g} = \Theta_X$ for smooth X, it is not clear whether there is any dimension estimate of the characteristic variety that can be deduced from this result. The main difficulty is that on the space $\mathbf{Spec}(S^\bullet(\mathfrak{g}))$ one does not have a symplectic structure so it makes no sense to speak about coisotropic subvarieties and one cannot conclude that $dim(char(\mathcal{M})) \geq dim(X)$. For the same reasons, the proof of the fact that the homological dimension of the ring $\mathcal{D}_{\mathbb{C}^n,0}$ equals n does not immediately generalize to the rings $\mathcal{D}_{\mathfrak{g},0}$.

3.1.3 The de Rham complex

In the theory of ordinary $\mathcal{D}_X$-modules (for X smooth) we can associate to any $\mathcal{D}_X$-module $\mathcal{M}$ its *de Rham*-complex, which generalizes the de Rham complex of differential forms. A similar construction exists for modules over general Lie algebroids. We start with a slightly more general situation by considering a Lie algebroid $\mathfrak{g}$ over X/S, an $\mathcal{O}_X$-module $\mathcal{M}$ and a morphism of $\mathcal{O}_X$-modules $\beta : \mathfrak{g} \to \mathfrak{c}_X(M)$. Denote $\mathcal{D}_{\mathfrak{g}}$ by $\mathcal{D}$ for short.

Definition 3.8. *Set $C^p(\mathfrak{g}, \mathcal{M}) := \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^p \mathfrak{g}, \mathcal{M})$ and define a differ-*

ential $\delta : \mathcal{C}^p(\mathfrak{g}, \mathcal{M}) \to \mathcal{C}^{p+1}(\mathfrak{g}, \mathcal{M})$*:*

$$\begin{aligned}&(\delta(\phi))(h_1 \wedge \ldots \wedge h_{p+1}) := \\ &\textstyle\sum_{i=1}^{p+1} (-1)^i \beta(h_i)\, \phi\left(h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_{p+1}\right) \\ &+ \sum_{1 \le i < j \le p+1} (-1)^{i+j-1} \phi\left([h_i, h_j] \wedge h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge h_{p+1}\right)\end{aligned}$$

Lemma 3.9. *If $\mathcal{M}$ is a $\mathfrak{g}$-module, i.e. if β is a morphism of Lie algebroids, then $\delta^2 = 0$ and we call the complex $(\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{M}), \delta)$ defined in this way the de Rham complex of the Lie algebroid $\mathfrak{g}$ with values in $\mathcal{M}$. Moreover, if $\mathfrak{g}$ is $\mathcal{O}_X$-projective, this complex can be canonically identified with $\mathcal{RH}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{M})$.*

Proof. Consider the following left $\mathcal{D}$-module:

$$Sp_{\mathcal{D}}^p := \mathcal{D} \otimes_{\mathcal{O}_X} \bigwedge^p \mathfrak{g}$$

with the map $s : Sp_{\mathcal{D}}^p \to Sp_{\mathcal{D}}^{p-1}$

$$\begin{aligned}&s(P \otimes h_1 \wedge \ldots \wedge h_p) := \\ &\textstyle\sum_{i=1}^{p} (-1)^i Ph_i \otimes h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_{p+1} + \\ &\sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes [h_i, h_j] \wedge h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge h_p\end{aligned}$$

The terminology is chosen according to ordinary $\mathcal{D}_X$-module theory for smooth X: in that case there is a resolution on the left $\mathcal{D}_X$-module $\mathcal{O}_X$ called *Spencer* complex which is defined just as above in our more general setting.

One has first to check that the map s is well defined, then one calculates its square. Both of these calculations are quite nasty but straightforward. We conclude that $(Sp_{\mathcal{D}}^\bullet, s)$ is a complex. This already suffices to prove the first statement of the lemma: If $\mathcal{M}$ is a $\mathcal{D}$-module, then we can apply the functor $\mathcal{H}om_{\mathcal{D}}(-, \mathcal{M})$ to the Spencer complex (in our extended sense). But obviously

$$\mathcal{H}om_{\mathcal{D}}(Sp_{\mathcal{D}}^p, \mathcal{M}) = \mathcal{H}om_{\mathcal{D}}(\mathcal{D} \otimes_{\mathcal{O}_X} \bigwedge^p \mathfrak{g}, \mathcal{M}) \cong \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^p \mathfrak{g}, \mathcal{M})$$

and the differential δ of the de Rham complex is the dual of the differential s from the Spencer complex under the functor $\mathcal{H}om_{\mathcal{D}}(-, \mathcal{M})$.

For the second statement, one needs to show that $Sp^{\bullet}_{\mathcal{D}}$ is a resolution of $\mathcal{O}_X$ (viewed as a $\mathcal{D}$-module) in case that $\mathfrak{g}$ is $\mathcal{O}_X$-projective. This is first proved in the case that $\mathfrak{g}$ is locally free over $\mathcal{O}_X$, just like in ordinary $\mathcal{D}$-module theory (see [Meb89]), namely, one considers a filtered version of the Spencer complex and deduces the acyclicity from the exactness of its associated graded complex, which is in fact a Koszul complex of the generators of $\mathfrak{g}$. The general case where $\mathfrak{g}$ is only $\mathcal{O}_X$-projective can then be deduced from this more special one. All these arguments are explained in detail in [Rin63]. □

Consider now the special case where $\mathcal{M} = \mathcal{O}_X$ with its natural structure of a left $\mathfrak{g}$-module mentioned above. $\mathcal{O}_X$ is an algebra, this allows us to construct a (graded) algebra structure on the complex $\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{O}_X)$ similar to the product of differential forms.

Definition 3.10. *Denote by* $\wedge$ *the following product:*

$$\begin{array}{rcl} \mathcal{C}^p(\mathfrak{g}, \mathcal{O}_X) \times \mathcal{C}^q(\mathfrak{g}, \mathcal{O}_X) & \longrightarrow & \mathcal{C}^{p+q}(\mathfrak{g}, \mathcal{O}_X) \\ (\Phi, \Psi) & \longmapsto & \Phi \wedge \Psi \end{array}$$

with

$$(\Phi \wedge \Psi)(f_1 \wedge \ldots \wedge f_{p+q}) =$$

$$\sum_{\substack{I \coprod J = \{1,\ldots,n\} \\ i_1<\ldots<i_p,\, j_1<\ldots<j_q}} sgn(I,J) \cdot \Phi(f_{i_1} \wedge \ldots \wedge f_{i_p}) \cdot \Psi(f_{j_1} \wedge \ldots \wedge f_{j_q})$$

The sign is defined as

$$sgn(I,J) := sgn\begin{pmatrix} 1, \ldots\ldots\ldots\ldots, p+q \\ i_1, \ldots, i_p, j_1, \ldots, j_q \end{pmatrix}$$

Theorem 3.11. *The triple* $(\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{O}_X), \delta, \wedge)$ *is a differential graded algebra (see definition A.2 on page 136). More precisely, we have for any* $\Phi \in \mathcal{C}^p$, $\Psi \in \mathcal{C}^q$ *and* $\Gamma \in \mathcal{C}^r$*:*

1. $\Phi \wedge \Psi = (-1)^{deg(\Phi) \cdot deg(\Psi)} \cdot \Psi \wedge \Phi$

2. $(\Phi \wedge \Psi) \wedge \Gamma = \Phi \wedge (\Psi \wedge \Gamma)$

3. $\delta(\Phi \wedge \Psi) = \delta(\Phi) \wedge \Psi + (-1)^{deg(\Phi)} \cdot \Phi \wedge \delta(\Psi)$

Proof. The first two points are trivial, while the third has to be checked by an explicit calculation. □

Up to this point, we have developed the theory of Lie algebroids in some analogy to ordinary $\mathcal{D}$-module theory. In particular, the complex $\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{M})$ is a generalization of the de Rham complex of a $\mathcal{D}$-module. The case $\mathcal{M} = \mathcal{O}_X$ is rather trivial in $\mathcal{D}$-module theory, it gives the usual de Rham complex of the manifold X. However, if X is singular, then there is the de Rham complex of Kähler differential forms (see the definition on page 18), which contains important information on the structure of the singularities. The complex $\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{O}_X)$ is related to the complex of Kähler differentials as the following lemma shows. Note that we have to consider the complex $\Omega^\bullet_{X/S}$ of relative differential forms.

Lemma 3.12. *Consider a Lie algebroid $\mathfrak{g}$ over X/S. Then there is a morphism of differential graded algebras $J : \Omega^\bullet_{X/S} \to \mathcal{C}^\bullet(\mathfrak{g}, \mathcal{O}_X)$.*

Proof. First we dualize the structure morphism $\alpha : \mathfrak{g} \to \Theta_{X/S}$ to get

$$\alpha^* : \left(\Omega_{X/S}\right)^{**} \to \mathfrak{g}^* = \mathcal{C}^1(\mathfrak{g}, \mathcal{O}_X)$$

Then we define J to be the composition $\alpha^* \circ \iota$, where $\iota : \Omega_{X/S} \to \left(\Omega_{X/S}\right)^{**}$ is the canonical morphism. The previously defined product structure on $\mathcal{C}^\bullet(\mathfrak{g}, \mathcal{O}_X)$ allows us to define an extension of J to the whole de Rham complex by setting

$$J(\omega_1 \wedge \ldots \wedge \omega_p) := J(\omega_1) \wedge \ldots \wedge J(\omega_p)$$

This shows directly that the morphism J is a morphism of graded algebras. But it is even a morphism of DGA's: It suffices to verify that the diagram

$$\begin{array}{ccc} & & \Omega^1_{X/S} \\ & \overset{d}{\nearrow} & \downarrow J \\ \mathcal{C}^0(\mathfrak{g}, \mathcal{O}_X) = \mathcal{O}_X = \Omega^0_{X/S} & & \\ & \underset{\delta}{\searrow} & \\ & & \mathcal{C}^1(\mathfrak{g}, \mathcal{O}_X) \end{array} \tag{3.1}$$

is commutative. This is obvious. □

3.2 The lagrangian Lie algebroid

After these generalities, we return to lagrangian singularities. We associate a Lie algebroid to any family of lagrangian subvarieties $\mathcal{L} \to S$ and consider its de Rham complex with coefficients in $\mathcal{O}_{\mathcal{L}}$. So let us be given a flat family

$$\begin{array}{ccc} \mathcal{L} & \hookrightarrow & M \times S \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle pr_2} & \\ S & & \end{array}$$

of lagrangian varieties over a base S. Recall that this means in particular that $\mathcal{L}$ is a reduced analytic subspace in the manifold $M \times S$, given by an ideal sheaf $\mathcal{I}$ such that $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$ ($\{\ ,\ \}$ is the Poisson structure on $M \times S$ induced by the symplectic form on M) and such that each fibre (one is sufficient) $\mathcal{L}_s$ has dimension n (where $dim(M) = 2n$).

Lemma 3.13. *Let $\mathfrak{g} := \mathcal{I}/\mathcal{I}^2$ be the conormal sheaf of $\mathcal{L}$. Then $\mathfrak{g}$ is a Lie algebroid on $\mathcal{L}/S$ which is isomorphic to $\Theta_{\mathcal{L}/S}$ on $(\mathcal{L}/S)_{reg}$, the regular locus of $f : \mathcal{L} \to S$.*

Proof. We have to define the Lie bracket and the structure morphism. The bracket is obviously induced by the Poisson bracket on $M \times S$, more precisely, we have $\{\mathcal{I}^i, \mathcal{I}^j\} \subset \mathcal{I}^{i+j-1}$ (this is rapidly verified by induction), thus there is a well defined bracket $\{\ ,\ \}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. By the same argument, there is a bracket $\mathfrak{g}\times\mathcal{O}_L\to\mathcal{O}_L$ such that $\{g,f\cdot h\}=\{g,f\}h+f\{g,h\}$ for $g\in\mathfrak{g}$ and $f,h\in\mathcal{O}_L$ and $\{g,f\}=0$ for $f\in\mathcal{O}_S$. This defines the structure morphism $\alpha:\mathfrak{g}\to\Theta_{\mathcal{L}/S}$ by setting $\alpha(g)=\{g,-\}$. α is a Lie algebra morphism, this follows immediately from the Jacobi identity in $\mathcal{O}_{M\times S}$.

Consider again the morphism $J:\Omega_{\mathcal{L}/S}\to(\mathcal{I}/\mathcal{I}^2)^*$ from above. This morphism is an isomorphism on $(\mathcal{L}/S)_{reg}$, because both $\Omega_{\mathcal{L}/S}$ and $(\mathcal{I}/\mathcal{I}^2)^*$ are locally free away from the singularities and can be identified with the sheaf of sections of the relative cotangent and conormal bundle. But these are canonically isomorphic because the regular locus of each fibre $\mathcal{L}_s$ is lagrangian in $M\times\{s\}$, see theorem 2.1 on page 52. Moreover, on $(\mathcal{L}/S)_{reg}$ we also have an isomorphism of $\Omega_{\mathcal{L}/S}$ with $(\Omega_{\mathcal{L}/S})^{**}$. This shows that the structure map is an isomorphism on $(\mathcal{L}/S)_{reg}$. □

Denote by $(C^\bullet_{\mathcal{L}/S},\delta,\wedge)$ the de Rham complex of the Lie algebroid $\mathfrak{g}$ with values in $\mathcal{O}_\mathcal{L}$ (with its DGA-structure). It is useful to write down explicitly the first terms of this complex:

$$\begin{array}{ccccc}
\mathcal{C}^0_{\mathcal{L}/S} & \longrightarrow & \mathcal{C}^1_{\mathcal{L}/S} & \longrightarrow & \mathcal{C}^2_{\mathcal{L}/S}\\
\| & & \| & & \|\\
\mathcal{O}_\mathcal{L} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_\mathcal{L}}(\mathfrak{g},\mathcal{O}_\mathcal{L}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_\mathcal{L}}(\bigwedge^2\mathfrak{g},\mathcal{O}_\mathcal{L})\\
 & & & & \\
h & \longmapsto & (f\mapsto\{f,h\}) & & \\
 & & & & \\
 & & \phi & \longmapsto & f_1\wedge f_2\mapsto\phi(\{f_1,f_2\})\\
 & & & & -\{f_1,\phi(f_2)\}-\{\phi(f_1),f_2\}
\end{array}$$

The product $\wedge:\mathcal{C}^0_{\mathcal{L}/S}\times\mathcal{C}^p_{\mathcal{L}/S}\to\mathcal{C}^p_{\mathcal{L}/S}$ is just the multiplication coming from the $\mathcal{O}_\mathcal{L}$-module structure on $\mathcal{C}^p_{\mathcal{L}/S}$, whereas

$$\begin{array}{rcl}
\bigwedge:\mathcal{C}^1_{\mathcal{L}/S}\times\mathcal{C}^1_{\mathcal{L}/S} & \longrightarrow & \mathcal{C}^2_{\mathcal{L}/S}\\
(\phi,\psi) & \longmapsto & (f_1\wedge f_2\mapsto\phi(f_1)\psi(f_2)-\phi(f_1)\psi(f_2))
\end{array}$$

Lemma 3.14. *The morphism $J:\Omega^\bullet_{\mathcal{L}/S}\to\mathcal{C}^\bullet_{\mathcal{L}/S}$ of DGA's is an isomorphism on $(\mathcal{L}/S)_{reg}$. Its kernel complex equals $\mathcal{T}ors(\Omega^\bullet_{\mathcal{L}/S})$ consisting of the torsion subsheaves of $\Omega^\bullet_{\mathcal{L}/S}$.*

Proof. $J^1 : \Omega^1_{\mathcal{L}/S} \to \mathcal{C}^1_{\mathcal{L}/S}$ was seen to be an isomorphism on $(\mathcal{L}/S)_{reg}$, this is obviously true for the whole morphism J.

The sheaves $\mathcal{C}^p_{\mathcal{L}/S}$ are of "Hom"-type, hence torsion free and therefore $\mathcal{T}ors(\Omega^\bullet_{\mathcal{L}/S})$ lies in the kernel of J. Conversely, any kernel element is torsion, because J is an isomorphism at a general point. □

We note a simple observation concerning the vanishing of the lagrangian de Rham complex.

Lemma 3.15. *Let $f : \mathcal{L} \to S$ a lagrangian family of relative dimension n and $x \in \mathcal{L}$ a point. Then the germ of $\mathcal{C}^p_{\mathcal{L},x}$ vanishes for all $p > n$.*

Proof. Let $L := \mathcal{L}_{f(x)}$ the fibre of f over $f(x)$. Then $\Omega^p_{L,x} = 0$ for all $p > n$ and $x \in L_{reg}$. Therefore, $\mathcal{C}^p_L$ is concentrated on the singular locus of L which is a proper subspace (L is reduced). But the sheaves $\mathcal{C}^p_L$ are torsion free, which leads immediately to $\mathcal{C}^p_{L,x} = 0$. □

3.3 Applications to deformation theory

Using the technical tools introduced so far, we state and prove our results on the deformation theory of lagrangian subvarieties. The main point is the description of the infinitesimal deformation space of a lagrangian singularity $(L, 0) \subset (M, 0)$ as the first cohomology of the complex $\mathcal{C}^\bullet_{L,0}$. However, it will also be of importance to consider the relative case, that is, the relative tangent space of the functor *LagDef* for a lagrangian family $\mathcal{L} \to S$. This is not more involved, therefore we treat directly this case, which includes the absolute one as usual (take $S = \{pt\}$). Given a lagrangian subvariety $L \subset M$, we conjecture that its infinitesimal deformations are controlled by the global lagrangian de Rham complex. This can be proved in some special cases.

3.3.1 Infinitesimal deformations

We consider a lagrangian family $f : \mathcal{L} \hookrightarrow M \times S \twoheadrightarrow S$ such that each fibre L_s is a small contractible representative of the germ $(L_s, 0) \subset (M \times \{s\}, (0, s))$ (which we suppose to be Stein for $\mathbb{K} = \mathbb{C}$).

Theorem 3.16. *The relative tangent space of* $LagDef_{\mathcal{L}/S}$ *is the sheaf* $f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$. *The cohomology in degree zero* $\mathcal{H}^0(\mathcal{C}^\bullet_{\mathcal{L}/S})$ *is* $f^{-1}\mathcal{O}_S$.

First we state a simple lemma, the proof of which can be found in [Ban94].

Lemma 3.17. *Let* U *be a symplectic manifold and suppose moreover that* $H^1(U,\mathbb{K}) = 0$ *(and that* U *is Stein in the complex case). Then the Lie algebra of the symplectomorphism group of* U *is exactly the Lie algebra of Hamiltonian vector fields on* U.

Proof of the theorem: We suppose that $\mathcal{L}$ is embedded in $U \times S$ where $U \subset M$ is a contractible (Stein) neighborhood of each fibre L_s in M.

We first proof the second statement. Take an element h which lies in $\mathcal{H}^0(\mathcal{C}^\bullet_{\mathcal{L}/S}) = \mathcal{K}er(\delta : \mathcal{O}_\mathcal{L} \to \mathcal{C}^1_{\mathcal{L}/S})$. Then $\{h, g\} \in \mathcal{I}$ for all $g \in \mathcal{I}$. If h is not constant on the fibres of f, then the ideal $(\mathcal{I}, h)$ is strictly larger than $\mathcal{I}$, not the whole ring and still involutive. This is a contradiction to the fact that $\mathcal{L}$ is a lagrangian family, which means that $\mathcal{I}$ is maximal under all involutive ideals. So the kernel must be the constant sheaf.

To prove that $\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S}) = T^1_{LagDef}(\mathcal{L}/S)$, two things have to be checked: As $\mathcal{C}^1_{\mathcal{L}/S}$ is the normal module of $\mathcal{L}$ in $U \times S$, we must first identify the elements of $\mathcal{K}er(\delta^1 : \mathcal{C}^1_{\mathcal{L}/S} \to \mathcal{C}^2_{\mathcal{L}/S})$ with the **lagrangian** deformations. Then we have to show that the image of $\delta^0 : \mathcal{O}_\mathcal{L} \to \mathcal{C}^1_{\mathcal{L}/S}$ are the trivial deformations. But this is easy, because for $f \in \mathcal{O}_\mathcal{L}$, $\delta(f)$ acts as H_f, thus inducing a trivial deformation of each fibre. Furthermore, of all deformations coming from relative vector fields on $M \times S$, only those induced by relative hamiltonian vector fields are trivial in the lagrangian sense (this follows from the preceding lemma).

Take an element $\Phi \in \mathcal{K}er(\delta^1)$, which means that

$$\phi\left(\{g, h\}\right) - \{g, \phi(h)\} - \{\phi(g), h\} = 0$$

for all $f, g \in \mathcal{I}/\mathcal{I}^2$. Then Φ corresponds to the deformation given by

$$\widetilde{\mathcal{I}} = (f_1 + \epsilon\phi(f_1), \ldots, f_k + \epsilon\phi(f_k))$$

The ideal $\widetilde{\mathcal{I}}$ is involutive iff for any two elements $f + \epsilon\phi(f), g + \epsilon\phi(g)$, we have $\{f + \epsilon\phi(f), g + \epsilon\phi(g)\} \in \widetilde{\mathcal{I}}$, which is equivalent to

$$F := \{f, g\} + \epsilon\left(\{f, \phi(g)\} + \{\phi(f), g\}\right) \in \widetilde{\mathcal{I}}$$

Consider $G := \{f,g\} + \epsilon\phi(\{f,g\})$, which is an element of $\widetilde{\mathcal{I}}$, so the condition $F \in \widetilde{\mathcal{I}}$ is equivalent to $F - G \in \widetilde{\mathcal{I}}$, that is

$$\{f, \phi(g)\} + \{\phi(f), g\} - \phi(\{f,g\}) \in \mathcal{I}$$

This means exactly that $\phi \in \mathcal{K}er(\delta^1)$. ☐

Given a family of lagrangian subvarieties $f : \mathcal{L} \hookrightarrow M \times S \twoheadrightarrow S$, one is of course interested in the global deformation spaces. We first observe the following

Corollary 3.18. *There is an exact sequence*

$$0 \longrightarrow R^1 f_* f^{-1}\mathcal{O}_S \longrightarrow \mathbb{R}^1 f_*(\mathcal{C}^\bullet_{\mathcal{L}/S}) \longrightarrow f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$$

$$\longrightarrow R^2 f_* f^{-1}\mathcal{O}_S \longrightarrow \mathbb{R}^2 f_*\mathcal{C}^\bullet_{\mathcal{L}/S}$$

Furthermore, there are two special cases:

- *Let the family $\mathcal{L}$ be contractible along the fibres of f. Then*

$$\mathbb{R}^1 f_*\mathcal{C}^\bullet_{\mathcal{L}/S} = f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$$

  and in fact: $T^1_{LagDef}(\mathcal{L}/S) = f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$.

- *Let f be smooth (and Stein if $\mathbb{K} = \mathbb{C}$). Then it follows that*

$$\mathbb{R}^1 f_*\mathcal{C}^\bullet_{\mathcal{L}/S} = R^1 f_* f^{-1}\mathcal{O}_S$$

  and the space of global deformations of the family $\mathcal{L} \to S$ is indeed $R^1 f_* f^{-1}\mathcal{O}_S$.

Proof. The exact sequence follows from the usual local to global spectral sequence. The assertion for a contractible family $\mathcal{L}$ is just the last theorem. In the second case, note that the space of embedded deformations is $f_*\mathcal{N}_{\mathcal{L}}$, where $\mathcal{N}_{\mathcal{L}}$ is the normal normal bundle of $\mathcal{L}$ in $M \times S$. Each fibre $\mathcal{L}_s$ is a smooth lagrangian submanifold of M, therefore we have a bundle isomorphism $\mathcal{N}_{\mathcal{L}} \cong \Omega_{\mathcal{L}/S}$. Therefore each infinitesimal deformation corresponds to a fibrewise global one-form on $\mathcal{L}$, i.e. a section of $f_*\Omega_{\mathcal{L}/S}$. It is closed iff the deformation is lagrangian and the subspace of exact one-forms are deformations induced by hamiltonian vector fields, these are precisely the trivial ones. This yields $T^1_{LagDef}(\mathcal{L}/S) = R^1 f_* f^{-1}\mathcal{O}_S$ (here the assumption that f is Stein is needed in the complex case). ☐

By analogy with the cotangent complex, we conjecture the following generalization.

Conjecture 3.19. *The space of infinitesimal lagrangian deformations of a family of analytic lagrangian subspaces $\mathcal{L} \subset M \times S$ is*

$$T^1_{LagDef}(\mathcal{L}/S) = \mathbb{R}^1 f_* \mathcal{C}^\bullet_{\mathcal{L}/S}$$

3.3.2 Obstructions

Unfortunately, the complex $(\mathcal{C}^\bullet_L, \delta)$ does not have a bracket, i.e. there is no controlling dg-Lie algebra for the functor $LagDef_L$. However, we can extract some information on the obstruction theory for this functor from the second cohomology of $\mathcal{C}^\bullet_L$. As there are only partial results on the obstruction theory, we restrict in what follows to the case of a (single) lagrangian germ $(L, 0) \subset (M, 0)$.

Theorem 3.20. *Chose for a given deformation $\Phi \in \mathcal{C}^1_L$ elements $g_i \in \mathcal{O}_M$ such that the class of g_i modulo $\mathcal{I}$ equals $\Phi(f_i)$. Denote by $ob_{f_i \wedge f_j}$ the class of the element $\{g_i, g_j\}$ in $\mathcal{O}_L$. Then we have the following: If there exists a map $ob : \mathcal{C}^1_L \to \mathcal{C}^2_L$ such that $ob(\Phi)(f_i \wedge f_j) = ob_{f_i \wedge f_j}$ then*

- $\delta\left(\mathcal{I}m(ob)\right) = 0$ *and* $ob\left(\mathcal{I}m(\delta : \mathcal{O}_L \to \mathcal{C}^1_L)\right) = 0$, *so ob defines a map*

$$ob : \mathcal{H}^1(\mathcal{C}^\bullet_L) \longrightarrow \mathcal{H}^2(\mathcal{C}^\bullet_L)$$

- $ob(\Phi) = 0 \in \mathcal{H}^2(\mathcal{C}^\bullet_L)$ *iff there exits a (not necessarily flat) deformation over* $Spec(\mathbb{K}[\epsilon]/\epsilon^3)$ *whose fibers are lagrangian subvarieties inducing the given deformation over* $Spec(\mathbb{K}[\epsilon]/\epsilon^2)$.

Proof. The first statement can be verified by a direct calculation which uses several times the Jacobi identity. So we suppose that there is a map $ob : \mathcal{H}^1(\mathcal{C}^\bullet_L) \to \mathcal{H}^2(\mathcal{C}^\bullet_L)$. Let $\Phi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ be an element of $\mathcal{K}er(ob)$. This condition is equivalent to the existence of $\Psi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ with $ob(\Phi) = \delta(\Psi)$, i.e.

$$\{\Phi(f), \Phi(g)\} = \Psi\left(\{f, g\}\right) - \{f, \Psi(g)\} - \{\Psi(f), g\} \qquad \forall f, g \in \mathcal{L}$$

But this means that the following ideal is involutive:

$$J = (f_1 + \epsilon\Phi(f_1) + \epsilon^2\Psi(f_1), \ldots, f_k + \epsilon\Phi(f_k) + \epsilon^2\Psi(f_k))$$

proving that the given lagrangian deformation can be lifted to third order. □

Remark: Unfortunately, the Poisson-bracket does not descend to $\mathcal{O}_L$, so it is not clear whether the elements $ob_{f_i \wedge f_j}$ always extend to a map $ob : \mathcal{C}^1_L \to \mathcal{C}^2_L$. Furthermore, $\mathcal{H}^2(\mathcal{C}^\bullet_L)$ does not contain any information on whether a given $\Phi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ can be lifted as a *flat* deformation. For these reasons, the last result is rather weak and of no great use in practical calculations. As already said, there is for the moment no complete obstruction theory for the functor $LagDef_L$. Meanwhile, we can give a condition for the T^1-lifting criterion to hold true.

Theorem 3.21. *Let $L \subset M$ be lagrangian and suppose that the functor Def_L is smooth and that $\mathcal{H}^2(\mathcal{C}^\bullet_L) = 0$. Then the T^1-lifting criterion holds for the functor $LagDef_L$, i.e., the functor is smooth in this case.*

Proof. We start by considering the functors Def_L and $EmbDef_L$. The latter is the functor of embedded deformations of L as an analytic space. It is a classical result that the natural transformation $EmbDef_L \to Def_L$ is smooth (see, e.g., [Art76]). Hence, for Def_L smooth we get that also $EmbDef_L$ is smooth.

Denote as usual by A_k the ring $\mathbb{K}[\epsilon]/\epsilon^{k+1}$ and by $\mathcal{L}_k$ a family of lagrangian varieties over A_k with zero fibre L, that is $\mathcal{L}_k \in LagDef_L(A_k)$. The relative tangent space $T^1_{EmbDef}(\mathcal{L}_k/A_k)$ (for $\mathcal{L}_k$ seen as lying in $EmbDef_L(A_k)$) equals

$$\mathcal{C}^1_{\mathcal{L}_k/A_k} = \mathcal{H}om_{\mathcal{O}_\mathcal{L}}(\mathcal{I}_k/\mathcal{I}_k^2, \mathcal{O}_\mathcal{L})$$

where $\mathcal{I}_k$ is the defining ideal sheaf of $\mathcal{L}_k$ in $\mathcal{O}_M \widehat{\otimes} A_k$. Now fix a non-negative integer n. What we need to prove is that for any given $\mathcal{L}_n \in LagDef_L(A_n)$ there exists an element in $LagDef_L(A_{n+1})$ which restricts to $\mathcal{L}_n$. We have from theorem 3.16 on page 72 that $T^1_{LagDef}(\mathcal{L}_n/A_n) = \mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}_n/A_n})$. The sequence

$$0 \longrightarrow \mathbb{K} \xrightarrow{\cdot \epsilon^n} A_n \longrightarrow A_{n-1} \longrightarrow 0$$

yields by tensoring with the flat A_n-module $\mathcal{O}_{\mathcal{L}_n}$

$$0 \longrightarrow \mathcal{O}_L \xrightarrow{\cdot \epsilon^n} \mathcal{O}_{\mathcal{L}_n} \longrightarrow \mathcal{O}_{\mathcal{L}_{n-1}} \longrightarrow 0 \tag{3.2}$$

Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathcal{L}_n}}\left(\bigwedge^{\bullet}\mathcal{I}_n/\mathcal{I}_n^2, -\right)$ to this sequence yields the exact sequence of complexes

$$0 \longrightarrow \mathcal{C}_L^{\bullet} \longrightarrow \mathcal{C}_{\mathcal{L}_n/A_n}^{\bullet} \longrightarrow \mathcal{C}_{\mathcal{L}_{n-1}/A_{n-1}}^{\bullet}$$

It is not exact on the right in general. However, using lemma A.22 on page 155 it follows that the T^1-lifting theorem holds for the functor $EmbDef_L$, so that the map $\mathcal{C}^1_{\mathcal{L}_n/A_n} \to \mathcal{C}^1_{\mathcal{L}_{n-1}/A_{n-1}}$ is surjective. Therefore, we obtain a connecting homomorphism and the following long exact cohomology sequence

$$\longrightarrow \mathcal{H}^1(\mathcal{C}^{\bullet}_{\mathcal{L}_n/A_n}) \longrightarrow \mathcal{H}^1(\mathcal{C}^{\bullet}_{\mathcal{L}_{n-1}/A_{n-1}}) \longrightarrow \mathcal{H}^2(\mathcal{C}^{\bullet}_L)$$

$$\longrightarrow \mathcal{H}^2(\mathcal{C}^{\bullet}_{\mathcal{L}_n/A_n}) \longrightarrow \mathcal{H}^2(\mathcal{C}^{\bullet}_{\mathcal{L}_{n-1}/A_{n-1}})$$

By assumption, $\mathcal{H}^2(\mathcal{C}^{\bullet}_L) = 0$ so we get a surjection

$$T^1_{\mathcal{L}_n/A_n} \twoheadrightarrow T^1_{\mathcal{L}_{n-1}/A_{n-1}}$$

Then the T^1-lifting criterion (theorem A.20 on page 153) yields the smoothness of $LagDef_L$. □

Corollary 3.22. *Let $L \subset M$ be either a complete intersection of arbitrary dimension or a Cohen-Macaulay surface. Then Def_L is smooth, in particular, $LagDef_L$ is smooth if $\mathcal{H}^2(\mathcal{C}^{\bullet}_L) = 0$.*

Proof. In both cases it is known that T^2_L (see corollary A.32 on page 169 for its definition) vanishes, which gives the smoothness of Def_L. □

We remark that is is not clear in which cases this theorem applies, because for smoothable lagrangian singularities it is likely that the dimension of $\mathcal{H}^2(\mathcal{C}^{\bullet}_L)$ equals the second Betti number of a smooth fibre (see corollary 3.40 on page 94), which might not vanish, at least for surfaces. However, vanishing of $\mathcal{H}^2(\mathcal{C}^{\bullet}_L)$ is not really needed in the proof, it suffices that the map

$$\mathcal{H}^2(\mathcal{C}^{\bullet}_L) \longrightarrow \mathcal{H}^2(\mathcal{C}^{\bullet}_{\mathcal{L}_n/A_n})$$

given by multiplication with ϵ^n is injective. This is a much weaker condition which hopefully can be verified for interesting classes of examples like complete intersection of codimension two Cohen-Macaulay spaces.

3.3.3 Stability of families

Up to now, we were only concerned with deformations over Artin bases. Therefore, all statements on versality were in fact statements on formal versality (existence of a hull, see definition A.6 on page 141). Indeed, very little is known about the existence of deformations over convergent bases which are semi-universal in the strong sense, i.e., when there exist *convergent* base changes which induces every given deformation. This has to be compared to the general situation in deformation theory, e.g. flat deformations of singularities, where one needs supplementary effort and rather different techniques to obtain the existence of semi-universal deformations (see [dJP00]). However, there is a result, due to M. Garay ([Gar02]) for the functor $LagDef_L$ which can be used to prove rigidity (in the analytic sense) for certain examples. In the quoted paper, the theorem is stated for complete intersections, but this assumption is not essential. We adopt the proof to the general case. In order to do this, we first introduce an important tool form general deformation theory in our setting, namely, the so-called Kodaira-Spencer map.

Lemma 3.23. *Consider any lagrangian family $\mathcal{L} \to S$. Then there is a natural morphism*

$$KS : \Theta_S \longrightarrow f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$$

called the Kodaira-Spencer map. We can also consider the so-called reduced Kodaira-Spencer map $KS_{red} : T_{S,0} \longrightarrow H^1(\mathcal{C}^\bullet_{L,0})$ (where $L := f^{-1}(0)$) which is the reduction of KS by the maximal ideal $\mathbf{m}_{\mathcal{O}_{S,0}}$. Then if KS_{red} is surjective then also KS is surjective.

Proof. The proof relies on the coherence of the relative cohomology sheaves of $\mathcal{C}^\bullet_{\mathcal{L}/S}$ for a lagrangian family. We defer the statement and the proof of this result to the next section (theorem 3.35 on page 88).

Let us first define the map KS. Denote by $\mathcal{I} \subset \mathcal{O}_{M\times S}$ the defining ideal sheaf of $\mathcal{L}$. Then we let $KS(\vartheta)$ be the class $[\Phi]$ in $\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ of the homomorphism

$$\Phi \in \mathcal{C}^1_{\mathcal{L}/S} = \mathcal{H}om_{\mathcal{O}_\mathcal{L}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_\mathcal{L})$$

defined by $\Phi(g) := \vartheta(g)$ for $g \in \mathcal{I}$ where ϑ is seen as a vector field in $\Theta_{M\times S}$. It is easily shown that Φ lies in the kernel of $\delta : \mathcal{C}^1_{\mathcal{L}/S} \to \mathcal{C}^2_{\mathcal{L}/S}$ because the Poisson bracket on $\mathcal{O}_{M\times S}$ and derivation with respect to S

commutes. From theorem 3.35 we know that $f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ is $\mathcal{O}_S$-coherent. Hence KS is a morphism between coherent $\mathcal{O}_S$-modules. Therefore, if the reduction modulo $\mathbf{m}_{\mathcal{O}_{S,0}}$ is surjective, the map KS is itself surjective. □

Now we state the theorem on stability of lagrangian families.

Theorem 3.24. *Let $(L,0) \subset (M,0)$ be given with $dim(H^1(\mathcal{C}^\bullet_{L,0})) < \infty$. Suppose that there is a flat lagrangian deformation $f : \mathcal{L} \to S$ over a smooth complex space S which is infinitesimal versal, i.e., such that the reduced Kodaira-Spencer map*

$$KS_{red} : T_0S \to H^1(\mathcal{C}^\bullet_{L,0})$$

is surjective. Then this family is stable, that is, each one-parameter deformation over a smooth base T is analytically (symplectic) equivalent to f.

For the proof, we need the general principle of integration of vector fields, which can be stated as follows.

Lemma 3.25. *Let $(X,0)$ be a germ of an analytic space and $\vartheta \in \Theta_{X,0}$ a derivation of $\mathcal{O}_{X,0}$ such that there exists $g \in \mathbf{m}_{\mathcal{O}_{X,0}}$ with $\vartheta(g) = 1 \in \mathcal{O}_{X,0}$. Then $K := ker(\vartheta)$ is an analytic subalgebra of $\mathcal{O}_{X,0}$ and the map $K\{G\} \to \mathcal{O}_{X,0}$, $G \mapsto g$ is an isomorphism.*

See the first chapter of [BF00] for the proof.

Proof of the theorem. Let $F : \mathcal{L}_T \to S_T$ be a one-parameter deformation of f over T, where T is an open neighborhood of the origin in $\mathbb{K}$. It follows from the last lemma that in order to show that the family F is trivial we have to find a compatible pair of vector fields $(\theta, \delta) \in \Theta_{\mathcal{L}_T} \times \Theta_{S_T}$ trivializing F. This means that $dF(\theta) = \delta$ and that there is a function $t \in \mathcal{O}_{S_T}$ with $\delta(t) = 1$ and $(F^*t)^{-1}(0) \cong \mathcal{L}$. The spaces S and T are smooth, therefore we can suppose that $S_T \cong S \times T$ and that $\mathcal{O}_{T,0} = \mathbb{K}\{t\}$. Denote by $p : S \times T \to S$ the projection and let $\mathcal{I}_T$ be the ideal which defines $\mathcal{L}_T$ in $M \times S \times T$. We are left to show that there is $\vartheta \in p^*\Theta_{M\times S}$ such that $(\vartheta + \partial_t)(\mathcal{I}_T) \subset \mathcal{I}_T$ (then $\vartheta + \partial_t$ defines the field $\theta \in \Theta_{\mathcal{L}_T}$ as required). However, there is one additional condition on ϑ, namely, we need that $\vartheta - df(\vartheta)$ is an element of $\mathcal{H}am_{M\times S/S} \subset \Theta_{M\times S/S}$, the space of relative vector field which are fibrewise hamiltonian, otherwise the automorphism obtained by integration would not be symplectic.

From the last lemma we have two surjective morphisms

$$KS_{\mathcal{L}/S} : \Theta_S \longrightarrow f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$$
$$KS_{\mathcal{L}_T/S_T} : \Theta_{S\times T} \longrightarrow F_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}_T/S_T})$$

These are the Kodaira-Spencer maps of the families $f : \mathcal{L} \to S$ and $F : \mathcal{L}_T \to S_T$. They are both surjective because their reductions modulo the respective maximal ideals ($\mathbf{m}_{\mathcal{O}_S}$ and $\mathbf{m}_{\mathcal{O}_{S\times T}}$) coincide and are surjective by assumption. Moreover, the natural restriction morphism $\mathcal{C}^\bullet_{\mathcal{L}_T/S_T} \to \mathcal{C}^\bullet_{\mathcal{L}/S}$ induces a map $F_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}_T/S_T}) \to f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$. We compose it with the inclusion $f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S}) \hookrightarrow p^* f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ to obtain a morphism

$$\Phi : F_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}_T/S_T}) \longrightarrow p^* f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$$

The reduction of this morphisms is the identity on $\mathcal{H}^1(\mathcal{C}^\bullet_L)$, therefore Φ is an isomorphism by the coherence of the two cohomology sheaves. This gives a surjective morphism

$$p^*KS_{\mathcal{L}/S} : p^*\Theta_S \longrightarrow F_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}_T/S_T})$$

so that there is $\vartheta_1 \in p^*\Theta_S$ with $p^*KS_{\mathcal{L}/S}(\vartheta_1) = KS_{\mathcal{L}_T/S_T}(\partial_t)$. Looking at the definition of the Kodaira-Spencer map, this equality (recall that it is an equality in the cohomology of the complex $\mathcal{C}^\bullet_{\mathcal{L}_T/S_T}$) shows that there exists a function $h \in \mathcal{O}_{M\times S\times T}$ such that $(X_h + \vartheta_1 + \partial_t)(\mathcal{I}_T) \subset \mathcal{I}_T$. Therefore the vector field $\vartheta := \vartheta_1 + X_h$ satisfies the requirements. This finishes the proof. □

Note that in abstract deformation theory as described in Appendix A one can construct a Kodaira-Spencer map for any cofibred groupoid. For the cofibred category $\mathbf{LagDef}_L$, this general description reduces to the above definition. The abstract Kodaira-Spencer map sits in an exact sequence (see [BF00] for a detailed account) and it seems that the proof of the theorem just given can be directly deduced form the exactness of this sequence. However, in order to do this one has to consider a category over the category of local analytic rings (and not only over **Art**) in order to get the convergent stability theorem as stated above.

In general, we do not yet have a versality theorem for lagrangian singularities, but the above stability criterion allows us to detect whether a given lagrangian singularity is rigid in a rather weak sense.

Theorem 3.26. *Let $(L, 0) \subset (M, 0)$ be lagrangian with $H^1(\mathcal{C}^\bullet_{L,0}) = 0$. Then any deformation $L_S \hookrightarrow M \times S \twoheadrightarrow S$ where S is a smooth analytic space can be trivialized by an analytic symplectomorphism.*

Proof. As $H^1(\mathcal{C}^\bullet_{L,0})$ vanishes, the family $L \to \{pt\}$ is infinitesimal versal. Thus the last theorem implies that any deformation $L_S \to S$ can be trivialized. ☐

This gives not yet rigidity in the usual sense, because we assume the base of the family $L_S \to S$ to be smooth in order to apply the theorem. Therefore it is a priori possible that there are deformations over singular curves which cannot be analytically trivialized in the symplectic category. This gap is still to be filled.

3.3.4 Integrable systems

In this section we will construct a controlling dg-Lie algebra for deformations of integrable systems. Its definition is a special case of the lagrangian de Rham complex. However, its terms are modules on the whole symplectic manifold, which is the main reason for the existence of a Lie bracket making it into a dg-Lie algebra.

So let us consider an analytic mapping $F : M \to U$, where M is a $2n$-dimensional symplectic manifold and U is an open domain in $\mathbb{K}^n$. Therefore, F has components $F = (f_1, \ldots, f_n)$. Then the condition for this system to be completely integrable is $\{f_i, f_j\} = 0 \in \mathcal{O}_M$ (see page 35 and page 57). To associate a Lie algebroid to this situation, consider the graph of the mapping F:

$$\begin{array}{ccc} M & \overset{\Gamma}{\hookrightarrow} & M \times U \\ {\scriptstyle F}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ U & & \end{array}$$

Denote by $\mathcal{L}$ the image of Γ and let $\mathcal{I} \subset \mathcal{O}_{M\times U}$ the defining ideal sheaf. It is immediate that $\mathcal{I}$ is involutive with respect to the Poisson bracket $\{\ ,\ \}_U$ on $M \times U$. So we are in the general situation described above and $\mathcal{I}/\mathcal{I}^2$ is a Lie algebroid over the morphism $pr_2 : M \times U \to U$. We denote the corresponding de Rham complex by $\mathcal{C}_F^\bullet$. It is a complex of locally free sheaves on M (because the graph is smooth in $M \times U$). It can be explicitly written down.

Lemma 3.27. *The terms of the complex* $\mathcal{C}_F^\bullet$ *are*

$$\mathcal{C}_F^p \cong \mathcal{H}om_{\mathcal{O}_M}\left(\bigwedge^p \mathcal{I}/\mathcal{I}^2, \mathcal{O}_M\right) \cong \bigwedge^p \mathcal{O}_M$$

together with the following differential

$$\begin{array}{rcl} \delta : \mathcal{C}_F^p \cong \mathcal{O}_M^{\binom{n}{p}} & \longrightarrow & \mathcal{C}_F^{p+1} \cong \mathcal{O}_M^{\binom{n}{p+1}} \\ (g_{i_1\dots i_p})_{i_1<\dots<i_p} & \longmapsto & \left(\sum_{l=1}^{n}(-1)^l\{f_l, g_{j_1\dots\widehat{j_l}\dots j_{p+1}}\}\right)_{j_1<\dots<j_{p+1}} \end{array}$$

Moreover, the product structure of $\mathcal{C}_F^\bullet$ *is given by*

$$\begin{array}{ccccc} \mathcal{C}_F^p & \times & \mathcal{C}_F^q & \longrightarrow & \mathcal{C}_F^{p+q} \\ \big((g_{i_1\dots i_p})_{i_1<\dots<i_p} & , & (h_{j_1\dots j_q})_{j_1<\dots<j_p}\big) & \longmapsto & g \wedge h \end{array}$$

with

$$g \wedge h \quad := \sum_{\substack{I \coprod J = \{1,\dots,n\} \\ i_1<\dots<i_p\ ,\ j_1<\dots<j_q}} sgn(I,J)\cdot g_{i_1\dots i_p}\cdot h_{j_1\dots j_q}$$

Proof. The conormal module $\mathcal{I}/\mathcal{I}^2$ of $\mathcal{L}$ is generated by the classes of the functions $F_i := x_i - f_i \in \mathcal{O}_{M\times U}$ where x_i are coordinates in U. Then the statements of the lemma are immediate by using the fact that $\{c, F_i\} = \{c, f_i\}$ for any $c \in \mathcal{O}_{\mathcal{L}} \cong \mathcal{O}_M$. □

The complex $\mathcal{C}_F^\bullet$ differs in one point from the the complex $\mathcal{C}_{\mathcal{L}/S}^\bullet$ for a general lagrangian family: it consists of modules of homomorphisms into $\mathcal{O}_M$, which is not only an algebra but also a Lie algebra under the Poisson bracket. This is not the case for the complex $\mathcal{C}_{\mathcal{L}/S}^\bullet$ in general and allows us to define the structure of a dg-Lie algebra on $\mathcal{C}_F^\bullet$ by using exactly the same formula as for the product

$$[g,h] \quad := \sum_{\substack{I \coprod J = \{1,\dots,n\} \\ i_1<\dots<i_p\ ,\ j_1<\dots<j_q}} sgn(I,J)\cdot \left\{g_{i_1\dots i_p}, h_{j_1\dots j_q}\right\}$$

Theorem 3.28. *The complex $(\mathcal{C}_F^\bullet, \delta, [\ ,\ ])$ has the structure of a dg-Lie algebra.*

Proof. One has to do the same explicit calculations as for the proof of theorem 3.11 on page 67. □

We can now describe the relation between the functor $HamDef_F$ and the complex $C_F^\bullet$ for a germ of a completely integrable system $F : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^n, 0)$. We consider the functor $HamDef_F$ as well as the dg-Lie algebra $\mathcal{C}_F^\bullet$ for a representative $F : V \to U$ with V and U open domains in M and $\mathbb{K}^n$, respectively.

Theorem 3.29. *Denote by $L := (\mathcal{C}_F^\bullet, \delta, [\ ,\ ])$ the dg-Lie algebra associated to the mapping F. Then there is an isomorphism of functors $\eta : Def_L \to HamDef_F$.*

Proof. The definition of the transformation η is straightforward: Let A be in **Art** and $\mathbf{g} \in MC_L(A)$, i.e.:

$$\mathbf{g} = (g_1, \dots, g_n) \in \mathcal{C}_F^1 \otimes \mathbf{m}_A = (\mathcal{O}_V \otimes \mathbf{m}_A)^n$$

such that $\delta(\mathbf{g}) + \frac{1}{2}[\mathbf{g}, \mathbf{g}] = 0$. This means that for any $i < j$

$$\{g_i, f_j\} - \{f_i, g_j\} + \{g_i, g_j\} = 0$$

which is easily seen to be equivalent to the vanishing of all commutators of the deformed system

$$F_S = (f_1 + g_1, \dots, f_n + g_n) : V \times Spec(A) \to U$$

On the other hand, each deformation of F, that is, an element in the groupoid $\mathbf{HamDef}_F(S)$ representing a given isomorphy class in the set $HamDef_F(S)$ is of the above form with all commutators vanishing. Therefore it defines an element in $MC_L(A)$. It remains to identify the group of S-symplectomorphisms with $G_L(S)$. But this is clear, as locally (see lemma 3.17 on page 72) each symplectomorphism is generated by a hamiltonian field and the action

$$\left(L^0 \otimes \mathbf{m}_S\right) \times \left(L^1 \otimes \mathbf{m}_S\right) \to L^1 \otimes \mathbf{m}_S$$

is precisely the action of a (relative) hamiltonian field on the deformed map F_S. □

3.4 Properties of the lagrangian de Rham complex

3.4.1 Constructibility and Coherence

As we have seen in the last chapter, the cohomology of the complex $\mathcal{C}^\bullet$ plays an important role in the deformation theory of L. From Schlessinger's theorem (theorem A.8 on page 142) we know that the main point in proving the existence of (formally) semi-universal deformations is the finiteness of this cohomology. The following section is devoted to study this problem. It turns out that there is a natural condition for a variety L that ensures that the cohomology of $\mathcal{C}^\bullet_L$ is finite over $\mathbb{K}$.

When one studies the functor Def_L of flat deformations (see section A.2.3), a formally (and even convergent) semi-universal deformation exists by Schlessinger's theorem if the singularities are isolated. Our condition therefore has to be seen as an analog (in the symplectic/lagrangian context) to the condition $dim(Sing(L)) = 0$. However, we insist on the fact that it is a considerably weaker condition, meaning that there are many lagrangian singularities with non-isolated singular locus having finite-dimensional T^1_{LagDef} (and eventually a semi-universal deformation). We will discuss examples in the next section.

In fact, we have two more precise results: First, in the absolute case, even to prove finiteness one needs to study the structure of the cohomology *sheaves* of the complex $\mathcal{C}^\bullet_L$. We will show that these cohomology sheaves are constructible with respect to a suitable stratification of the variety L. On the other hand, the complex $\mathcal{C}^\bullet$ has been introduced in a relative setting for a morphism $f : \mathcal{L} \to S$. In this case one is interested in the hyperdirect image sheaves $\mathbb{R}^i f_* \mathcal{C}^\bullet_{\mathcal{L}/S}$. The preceding result can be extended to prove the coherence of these sheaves. It is an open problem whether they are always free. However, for $i = 1$ freeness can be proved under some assumptions yielding a nice application for smoothable singularities L.

We start by introducing the above mentioned condition. In the whole section, we will consider a Stein representative L for a germ $(L, 0) \subset (M, 0)$ of a lagrangian singularity.

Definition 3.30. *Define S_k^L to be the following set*

$$S_k^L := \{p \in L \mid edim(p) = 2n - k\} \subset L$$

for $k \in \{0, \ldots, n\}$ where $edim(p)$ is the embedding dimension of the germ (L, p). Then we will say that L satisfies "Condition P" iff the inequality $dim(S_k^L) \leq k$ holds for all k.

The following lemma explains the meaning of this condition in somewhat more geometric terms.

Lemma 3.31. *Let $p \in S_k^L \subset L$ with $k > 0$. Then the germ (L, p) can be decomposed into a product*

$$(L, p) \cong (L', p') \times (C, 0)$$

where $(C, 0)$ is the germ of a smooth curve. This decomposition is compatible with the decomposition of the ambient symplectic space

$$(M, 0) \cong (M', 0) \times (M'', 0)$$

(with $dim(M) = 2n$, $dim(M') = 2n-2$ and $dim(M'') = 2$) by symplectic reduction. Therefore, L' is a lagrangian variety in the symplectic space M'. Furthermore, we have $p' \in S_{k-1}^{L'}$.

Proof. Recall theorem 1.5 on page 15: Any non-degenerate hamiltonian function on a symplectic manifold fibres (locally) its own level hypersurfaces in symplectic leafs. We have $k \geq 1$, therefore there is a non-degenerate hamiltonian function h on M which vanishes oh L, this implies that X_h is tangent to L. Then by integration of vector fields (lemma 3.25 on page 78 with $(X, 0) = (L, p)$), we get the required decomposition $(L, p) \cong (L', p') \times (C, 0)$ inside $(M, 0) \cong (M', 0) \times (M'', 0)$. Obviously, $edim(L', p') = edim(L, p) - 1$. □

This result implies that whenever a stratum S_k^L is non-empty then there are k independent non-degenerate hamiltonian vector fields defined in a neighborhood of a point $p \in S_k^L$ which are tangent to S_k^L. Thus, the dimension of this stratum must be at least k. So "Condition P" can be restated by saying that either $\dim(S_k^L) = k$ or $S_k^L = \emptyset$.

The preceding lemma can be used to show that there are germs of singular spaces which do not admit any lagrangian embedding.

Corollary 3.32. *Let $n > 1$ and $(X, 0) \subset (\mathbb{K}^{n+1}, 0)$ be an isolated hypersurface singularity. Then there is no lagrangian embedding $(X, 0) \hookrightarrow (\mathbb{K}^{2n}, 0)$.*

Proof. Suppose that a lagrangian embedding exists. The embedding dimension of the germ $(X, 0)$ is $n + 1 < 2n$, so by the previous lemma there is a decomposition $(X, 0) = (Y, 0) \times (\mathbb{K}^{n-1}, 0)$ showing that $(X, 0)$ has non-isolated singularities. □

The two preceding results can be found in [Giv88]. We show now that for products of a lagrangian germ with a smooth factor the deformation theory (and more generally the cohomology of the whole complex) behaves particularly well. The phenomenon described by the following lemma is illustrated in figure 3.1 on page 87.

Lemma 3.33 (Propagation of Deformations). *Let $(L, 0) \subset (M, 0)$ be a germ of a lagrangian subvariety which can be decomposed, i.e., there is a germ $(L', 0)$ (which is lagrangian in $(M', 0)$) such that $(L, 0) = (L', 0) \times (C, 0)$ with C a smooth curve. Denote by $\pi : L \to L'$ the projection. Then there is a quasi-isomorphism of sheaf complexes*

$$j : \pi^{-1}\mathcal{C}^\bullet_{L'} \to \mathcal{C}^\bullet_L$$

Proof. Let $h \in \mathcal{O}_{M,0}$ be the hamiltonian function which fibres M and L. Then there is (as it follows from the proof of lemma 3.31 and from lemma 3.25 on page 78) a function $g \in \mathbf{m}_{\mathcal{O}_{M,0}}$ with $\{h,g\} = 1$. Let $I \subset \mathcal{O}_{M,0}$ resp. $I' \subset \mathcal{O}_{M',0}$ define $(L,0)$ resp. $(L',0)$. Then

$$\mathcal{O}_{M',0} = \{\alpha \in \mathcal{O}_{H,0} \mid \{\alpha,h\} = 0\} \quad \text{and}$$

$$I' = I \cap \mathcal{O}_{M',0} = \{\alpha \in I \mid \{\alpha,h\} = 0\}$$

where H is the smooth hypersurface in M given by the vanishing of h. More specifically, we have $\mathcal{O}_{H,0} \cong \mathcal{O}_{M',0}\{g\}$ and $\mathcal{O}_{L,0} \cong \mathcal{O}_{L',0}\{g\}$. This implies the following relation between the conormal modules I/I^2 and I'/I'^2:

$$I/I^2 = \left(I'/I'^2 \otimes_{\mathcal{O}_{L',0}} \mathcal{O}_{L,0}\right) \oplus \mathcal{O}_{L,0}$$

where the (free) factor $\mathcal{O}_{L,0}$ is generated by the class of h in I/I^2. Furthermore, the Lie algebra structure on I/I^2 is of special type: For all $f_i \in I'/I'^2$ the bracket $\{h, f_i\}$ vanishes. We get

$$\bigwedge^p I/I^2 = \left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge^p I'/I'^2\right) \oplus \left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge^{p-1} I'/I'^2\right)$$

and

$$\begin{aligned} \mathcal{C}^p_{L,0} \quad = \quad & \mathit{Hom}_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \textstyle\bigwedge^p I'/I'^2, \mathcal{O}_{L,0}\right) \\ & \oplus \; \mathit{Hom}_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \textstyle\bigwedge^{p-1} I'/I'^2, \mathcal{O}_{L,0}\right) \end{aligned}$$

We write elements of $\mathcal{C}^p_{L,0}$ as infinite sums of type $\sum_{i=0}^{\infty} (\Phi_i, \Psi_i)\, g^i$. Then the differential is (for details of the calculation see [Sev99]):

$$\begin{array}{cccc} \delta : & \mathcal{C}^p_{L,0} & \longrightarrow & \mathcal{C}^{p+1}_{L,0} \\ & \sum\limits_{i=0}^{\infty} (\Phi_i, \Psi_i)\, g^i & \mapsto & \sum\limits_{i=0}^{\infty} \left(\delta\Phi_i, \delta\Psi_i + (-1)^{p+1}(i+1)\Phi_{i+1}\right) g^i \end{array}$$

We define the morphism j to be the inclusion

$$\begin{array}{ll} \mathit{Hom}_{\mathcal{O}_{L',0}}\left(\bigwedge^p I'/I'^2, \mathcal{O}_{L',0}\right) & \hookrightarrow \\ & \\ \mathit{Hom}_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge^p I'/I'^2, \mathcal{O}_{L,0}\right) & \oplus \\ \mathit{Hom}_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge^{p-1} I'/I'^2, \mathcal{O}_{L,0}\right) & \\ & \\ \Phi \longmapsto (\Phi, 0) \cdot g^0 & \end{array}$$

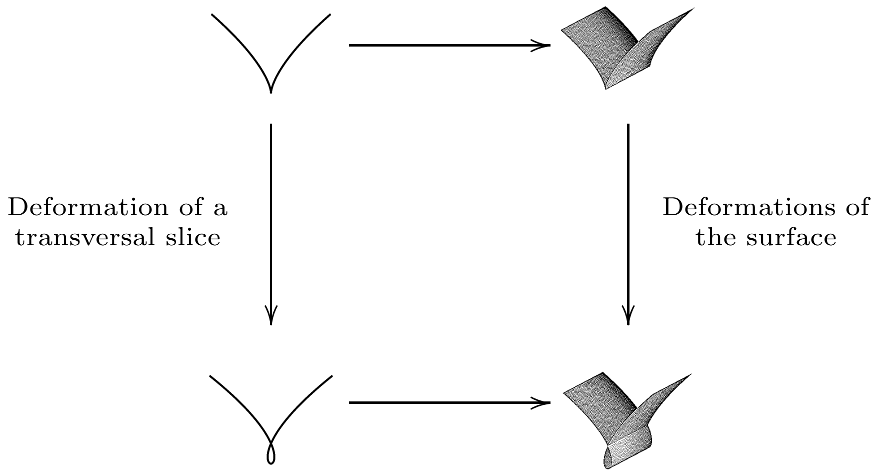


Figure 3.1: Propagation of deformations

It remains to show that the cokernel of this inclusion is acyclic. Then it follows immediately from the long exact cohomology sequence that j is a quasi-isomorphism. So let Γ be an element outside of the image of j such that $\delta(\Gamma) = 0$, that is:

$$\Gamma = \sum_{i=1}^{\infty} (\Phi_i, \Psi_i) g^i + (0, \Psi_0)$$

with $\delta\Phi_i = 0$ and $\delta\Psi_i = (-1)^p(i+1)\Phi_{i+1}$ for all $i \in \{0, 1, \ldots\}$. But then Γ vanishes in the cohomology because it can be written as $\Gamma = \delta\Lambda$ with

$$\Lambda := \sum_{i=1}^{\infty} \left(\frac{(-1)^p \Psi_{i-1}}{i}, 0 \right) g^i \in \mathcal{C}_{L,0}^{p-1}$$

□

Corollary 3.34. *There are isomorphisms of sheaves*

$$\pi^{-1}\mathcal{H}^i(\mathcal{C}_{L'}^\bullet) \cong \mathcal{H}^i(\mathcal{C}_L^\bullet)$$

Proof. This follows because π^{-1} is an exact functor. □

Now we come to the main theorem of this section. We consider directly the relative situation of a lagrangian family $\mathcal{L} \to S$. We restrict to the complex case for simplicity. If $\mathbb{K} = \mathbb{R}$, one might consider the complexification of the lagrangian variety. Recall that we suppose the morphism $f : \mathcal{L} \to S$ to be Stein.

Theorem 3.35. *Suppose that "Condition P" is satisfied for each fibre $\mathcal{L}_s$ of f. Then*

- $\mathcal{H}^i(\mathcal{C}^\bullet_{\mathcal{L}_s})$ *are constructible sheaves of finite dimensional vector spaces with respect to the stratification given by the* $S_k^{\mathcal{L}_s}$.
- $\mathbb{R}^i f_* \mathcal{C}^\bullet_{\mathcal{L}/S}$ *is a coherent sheaf of* $\mathcal{O}_S$*-modules.*
- $\mathbb{R}^i f_* \mathcal{C}^\bullet_{\mathcal{L}/S} = f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ *and moreover* $\left(\mathbb{R}^i f_* \mathcal{C}^\bullet_{\mathcal{L}/S}\right)_0 = H^i(\mathcal{C}_{\mathcal{L}/S,0})$

Proof. First note that by embedding $S \overset{i}{\hookrightarrow} U$ into a smooth ambient space U and by considering the higher direct image sheaves of the composition $i \circ f$, we can always assume S to be smooth.

For the first part, two things have to checked: We must prove that the restriction of the cohomology sheaves to the strata $S_k^{\mathcal{L}_s}$ are locally constant and that the stalks of $\mathcal{H}^i(\mathcal{C}^\bullet_{\mathcal{L}_s})$ are finite dimensional over $\mathbb{C}$. The first statement is a direct consequence of the last corollary: Let $p \in S_k^{\mathcal{L}_s}$ be a point at which $\mathcal{L}_s$ is decomposable, i.e. $k > 0$. By induction, we find a neighborhood $U \subset \mathcal{L}_s$ of p such there is an analytic isomorphism $h : U \xrightarrow{\cong} Z \times B_\epsilon^k$, where Z is lagrangian in M' with $dim(M') = 2n-2$, $B_\epsilon := \{z \in \mathbb{C} \,|\, |z| < \epsilon\}$ and each $q \in U \cap S_l^{\mathcal{L}_s}$ corresponds via h to a point $(q', b) \in Z \times B_\epsilon^k$ with $q' \in S_{l-k}^Z$. In particular, the image of $U \cap S_k^{\mathcal{L}_s}$ under h is $(\{pt\}, B(\epsilon)^k)$, so $\mathcal{H}^p(\mathcal{C}^\bullet_{\mathcal{L}_s})$ is constant on $U \cap S_k^{\mathcal{L}_s}$.

Now we come to the next part. In fact, in order to prove that the stalks of $\mathcal{H}^i(\mathcal{C}^\bullet_{\mathcal{L}_s})$ are finite dimensional it suffices to show that $\mathbb{R}^i f_* \mathcal{C}^\bullet_{\mathcal{L}}$ is coherent on S. In this way we can handle the two parts of the theorem at the same time. The following theorem, which we adapt from [vS87], will be used. Note that it uses the fundamental result from functional analysis (proposition B.10). For further references, see also [BG80] and the appendix of [Gar02].

Theorem 3.36. *Let a germ $g : (Y, 0) \to (T, 0)$ of a flat Stein morphism of complex spaces be given. Denote by $(Y_0, 0) := (g^{-1}(0), 0)$ the germ at zero of the zero fibre of g. Embed Y_0 and T in some $\mathbb{C}^N$ and in $\mathbb{C}^M$, respectively, so that $Y \subset \mathbb{C}^N \times T$. Choose a so called* ***standard representative*** *$g : X \to S$, i.e. a morphism representing the given germ such that:*

1. *$S := S_\eta := T \cap D_\eta$*
2. *$X := X_{\epsilon,\eta} := ((B_\epsilon \times S) \cap Y) \cap g^{-1}(D_\eta)$*

for an open ϵ-ball $B_\epsilon \subset \mathbb{C}^N$ and an open η-ball $D_\eta \subset \mathbb{C}^M$. For small ϵ and η the fibres of g will be Stein, contractible and intersect $\partial B_\epsilon \times S$ transversally. Let $(\mathcal{K}^\bullet, d)$ be a sheaf complex on X with the following properties

1. *all $\mathcal{K}^p$ are $\mathcal{O}_X$-coherent*
2. *the differentials $d : \mathcal{K}^p \to \mathcal{K}^{p+1}$ are $g^{-1}\mathcal{O}_S$-linear*
3. *there is a neighborhood U of $\overline{\partial X} := \overline{(\partial B_\epsilon \times S) \cap Y \cap g^{-1}(D_\eta)}$ in $\mathbb{C}^N \times S$ and a vector field ϑ of class C^∞ on U such that*
   - *ϑ is transversal to $\partial B_\epsilon \times S$*
   - *the flow of ϑ respects X and the fibers of g.*
   - *the restriction of the cohomology sheaves $\mathcal{H}^p(\mathcal{K}^\bullet)$ to the integral curves of ϑ are locally constant sheaves.*

Then the sheaves $\mathbb{R}^p g_* \mathcal{K}^\bullet$ are $\mathcal{O}_S$-coherent.

If we take $(Y, 0) = (\mathcal{L}, 0) \subset (M \times S, 0)$, $g = f$ and $\mathcal{K}^\bullet = \mathcal{C}^\bullet_{\mathcal{L}/S}$, then the only thing to verify is the existence of a vector field as described in the theorem. Choose a standard representative $X := \mathcal{L}_{\epsilon',\eta'} \to S := S_\eta$ such that on each fibre X_s we have $edim_{X_s}(p) < 2n$ for all $p \notin \{0\} \times S$ (this is possible due to "Condition P"). The vector field we are looking for will be constructed in two steps: Let $p \in \overline{\partial X}$ be a point with $p \in S_k^{X_s}$ with $k > 0$. Then it follows from lemma 3.31 on page 84 that there are k independent holomorphic hamiltonian vector fields $\eta_1, \ldots, \eta_k$ tangent to the stratum $S_k^{X_s}$ and to the fibres of f. We lift them to sections of $\Theta_{M \times S/S}$ defined in a neighborhood $U_p \subset M \times S$ of p. The stratum $S_k^{X_s}$ has complex dimension k ("Condition P"), therefore, the $2k$ fields $\eta_1, \ldots, \eta_k, i\eta_1, \ldots, i\eta_k$, viewed as C^∞-vector fields span the real tangent space of $S_k^{X_s}$ at p. As $S_k^{X_s}$ was transversal to $\overline{\partial X}_s := \overline{(\partial B_\epsilon \times \{s\}) \cap Y \cap g^{-1}(D_\eta)}$, a linear combination of these vector fields will define a field as required in the neighborhood U_p. Here we use the fact that the cohomology sheaves of $\mathcal{C}^\bullet_{\mathcal{L}/S}$ are constant on the strata $S_k^{X_s}$, thus on the integral curves of the above fields. To conclude, we choose a partition of unity subordinate to the covering of a neighborhood U of $\overline{\partial X}$ defined by the U_p. This allows us to glue the fields defined on the neighborhoods U_p to a field as required in theorem 3.36. Thus the coherence of the higher direct image sheaves is proved.

The last part of the theorem follows easily as in [vS87] (Proposition 1), because the vector field constructed above defines for each $s \in S$ a shrinking of $f^{-1}(s)$ onto one point as required in the proof of the proposition in loc.cit. □

Summarizing what has been done, we get the following main result by theorem 3.16 on page 72, theorem 3.35 on page 88 and Schlessinger's result (theorem A.8 on page 142).

Theorem 3.37. *Let $(L,0) \subset (M,0)$ be a lagrangian singularity satisfying "Condition P". Then there exists a formally semi-universal deformation $(L_S,0) \hookrightarrow (M \times S, 0) \twoheadrightarrow (S,0)$ with $S \in \widehat{\mathbf{Art}}$ which satisfies*

$$dim\left((\mathbf{m}_{\mathcal{O}_S}/\mathbf{m}^2_{\mathcal{O}_S})^*\right) = dim(H^1(\mathcal{C}^\bullet_{L,0}))$$

It is a very natural question to ask whether "Condition P" is always satisfied for a lagrangian variety $L \subset M$. This is obviously not the case for non-reduced spaces L, but the following example (which can be found in [Giv88], see also the discussion on page 21) shows that there exist even reduced varieties $L \subset M$ where points with maximal embedding dimension are non-isolated.

Consider any non quasi-homogeneous plane curve singularity $(C,0) \subset (\mathbb{C}^2,0)$. It has an associated legendrian space curve $K := Im(F,n)$, where $F \in \mathcal{O}_{\widetilde{C}}$ is the generating function and $n : \widetilde{C} \to C$ the normalization map. K is a singular legendrian subspace of the contact manifold $\mathbb{C}^3$. Now for any germ of a contact manifold $(K,0)$ of dimension $2n-1$

we can equip the direct product $(M,0) = (K,0) \times (\mathbb{C}^*, p)$ with a symplectic structure (which is called symplectization of $(K,0)$ in [Giv88]): in our example, if (p,q,z,t) are local coordinates on $M = \mathbb{C}^3 \times \mathbb{C}^*$ on $(M,0)$, then

$$\omega = d\,(t(dz - pdq))$$

We have the projection $\pi : (M,0) \to (K,0)$ and the preimage $L := \pi^{-1}(\Lambda)$ is a lagrangian subspace of $(M,0)$. Obviously, at all points $(0,q) \in L$ we have $edim_{(0,q)}L = 4$. Therefore, $(L,0)$ does not satisfy "Condition P". Probably, there are examples of this type where the cohomology of $\mathcal{C}^\bullet_{L,0}$ (and in particular the tangent space of $LagDef_{L,0}$) is not finite over $\mathbb{C}$. However, as these spaces are non-quasihomogenous, a direct calculation of the cohomology of the complex $\mathcal{C}^\bullet$ is very difficult (see section 3.5).

Remark: By the *Riemann-Hilbert-correspondence* ([Bjö93]), the complex $\mathcal{C}^\bullet_L$, viewed as an object of $\mathcal{D}^b_c(\mathbb{C}_M)$ (the derived category of constructible sheaves of $\mathbb{C}$-vector spaces on M) corresponds via the (inverse of the) *de Rham*-functor to a unique complex of coherent $\mathcal{D}_M$-modules with regular holonomic cohomology supported on L (i.e. an object of $\mathcal{D}^b_{\text{r.h.}}(\mu_L(\mathcal{D}_M))$).

Lemma 3.38. *Let $L \subset M$ satisfy "Condition P". Then the complex $\mathcal{C}^\bullet_L$ satisfies the first perversity condition, that is, the following inequality holds.*

$$\dim supp(\mathcal{H}^i(\mathcal{C}^\bullet_L)) \leq n - i$$

Proof. Let $p \in S^L_k$. Then $(L,p) = (L',p') \times (\mathbb{C}^k, 0)$ and $\mathcal{H}^i(\mathcal{C}^\bullet_L)_p = \mathcal{H}^i(\mathcal{C}^\bullet_{L'})_{p'}$. But $\dim(L') = n-k$, so $\mathcal{H}^i(\mathcal{C}^\bullet_{L'})_{p'} = 0$ for all $i > n-k$. This means that for fixed i, $\mathcal{H}^i(\mathcal{C}^\bullet_L)_p = 0$ for $p \in S^L_k$ for all $k > n-i$. So $\mathcal{H}^i(\mathcal{C}^\bullet_L)$ is supported on the strata S^L_k for $k \leq n-i$. By "Condition P" they are of dimension less or equal $n-i$. □

The second perversity condition means that

$$\dim supp(\mathcal{H}^i_V(\mathcal{C}^\bullet_L)) \leq \dim(V)$$

for any irreducible subspace $V \subset L$ and any $i \in \{0, \ldots, n - \dim(V)\}$. Here $\mathcal{H}^i_V(\mathcal{C}^\bullet_L)$ is the i-th local cohomology sheaf with respect to V of $\mathcal{C}^\bullet$.

It is not known whether this condition is always satisfied by a variety L with constructible complex $\mathcal{C}_L^\bullet$. Whenever this is the case, the $\mathcal{H}^i$'s are the *de Rham*-cohomology modules of a *single* $\mathcal{D}_M$-module supported on L. One might speculate that there is some operation (direct image in an appropriate category, like the modules over Lie algebroids) that produces this $\mathcal{D}_M$ module (or a whole complex in case that the cohomology of $\mathcal{C}_L^\bullet$ is not perverse) from the $\mathcal{D}_{\mathcal{I}/\mathcal{I}^2}$-module $\mathcal{O}_L$ and commutes with the de Rham functor.

3.4.2 Freeness of the relative cohomology

Now that we know about the coherence of the cohomology of $\mathcal{C}_{\mathcal{L}/S}^\bullet$, one might ask about its freeness. This is an open problem in general, but there is a partial result for the first cohomology. The ideas presented in this section can also be found in [GvS02].

Theorem 3.39. *Consider a lagrangian family $f : \mathcal{L} \to S$ over a smooth base S and suppose that*

- *$L := f^{-1}(0)$ is a complete intersection.*
- *The family is an infinitesimal miniversal deformation of L (in the sense of theorem 3.24 on page 78, i.e., the reduced Kodaira-Spancer map is an isomorphism). In particular, we have that $dim(S) = dim\left(H^1(\mathcal{C}_{L,0}^\bullet)\right)$.*

Then $f_*\mathcal{H}^1(\mathcal{C}_{\mathcal{L}/S}^\bullet)$ is a locally free sheaf of $\mathcal{O}_S$-modules. Moreover, if $dim(L) = 2$, then $f_*\mathcal{H}^2(\mathcal{C}_{\mathcal{L}/S}^\bullet)$ is also locally free.

Proof. We will show that the stalk of $f_*\mathcal{H}^1(\mathcal{C}_{\mathcal{L}/S}^\bullet)$ at zero (which we denote temporarily by H) is a free $\mathcal{O}_{S,0}$-module. We know from theorem 3.35 on page 88 that H is finitely generated and equals $H^1(\mathcal{C}_{\mathcal{L}/S,0}^\bullet)$. It will be sufficient to show that H is a Cohen-Macaulay module, that is, $depth(H) = dim(S)$. Denote $\mathcal{C}_{\mathcal{L}/S,0}^\bullet$ by $\mathcal{C}^\bullet$ for short and chose a system of parameters $(s_1, \ldots, s_k)$ of S. From the freeness of $\mathcal{I}/\mathcal{I}^2$ we get the existence of a short exact sequence of complexes

$$0 \to \frac{\mathcal{C}^\bullet}{(s_1, \ldots, s_i)\,\mathcal{C}^\bullet} \xrightarrow{\cdot s_{i+1}} \frac{\mathcal{C}^\bullet}{(s_1, \ldots, s_i)\,\mathcal{C}^\bullet} \longrightarrow \frac{\mathcal{C}^\bullet}{(s_1, \ldots, s_{i+1})\,\mathcal{C}^\bullet} \to 0$$

The long exact cohomology sequence yields

$$\begin{array}{rcl} \ldots \to H^0\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right) & \xrightarrow{\alpha} & H^0\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right) \\ \to H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right) & \xrightarrow{\cdot s_{i+1}} & H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right) \\ \to H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right) & \longrightarrow & \ldots \end{array}$$

But we have identifications $H^0\left(\mathcal{C}^\bullet/(s_1,\ldots,s_j)\mathcal{C}^\bullet\right) \cong \mathbb{C}\{s_{j+1},\ldots,s_k\}$ for any $j \in \{1,\ldots,k\}$, so the map α is just the restriction

$$\mathbb{C}\{s_{i+1},\ldots,s_k\} \longrightarrow \mathbb{C}\{s_{i+2},\ldots,s_k\}$$

which sends h to $h_{|s_{i+1}=0}$ and therefore surjective. This yields injectivity of

$$H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right) \xrightarrow{\cdot s_{i+1}} H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)$$

To conclude, we need to indentify the modules $H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)$ and $H^1(\mathcal{C}^\bullet)/\left(s_1,\ldots,s_i\right)H^1(\mathcal{C}^\bullet)$. The long exact sequence shows that the there is an inclusion

$$H^1(\mathcal{C}^\bullet)/\left(s_1,\ldots,s_{i+1}\right)H^1(\mathcal{C}^\bullet) \hookrightarrow H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right)$$

Consider the Kodaira-Spencer map $KS : \Theta_S \to H^1(\mathcal{C}^\bullet)$ of the family f (see lemma 3.23 on page 77 for its definition). Tensoring with $\mathcal{O}_S/(s_1,\ldots,s_{i+1})\mathcal{O}_S$ yields a morphism

$$KS_{i+1} : \Theta_S/(s_1,\ldots,s_{i+1})\Theta_S \longrightarrow H^1(\mathcal{C}^\bullet)/(s_1,\ldots,s_{i+1})H^1(\mathcal{C}^\bullet)$$

Compose it with the above inclusion to obtain a morphism

$$\Theta_S/(s_1,\ldots,s_{i+1})\Theta_S \longrightarrow H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right)$$

The reduction of this morphism modulo the ideal $(s_{i+1},\ldots,s_k)$ is the reduced Kodaira-Spencer map of the family f, therefore, it is surjective by assumption. Coherence of the two sheaves shows that the morphism itself is surjective. Now we have a commutative diagram

$$\begin{array}{ccc} \Theta_S/(s_1,\ldots,s_i)\Theta_S & \longrightarrow & H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right) \\ \downarrow & & \downarrow \\ \Theta_S/(s_1,\ldots,s_{i+1})\Theta_S & \twoheadrightarrow & H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right) \end{array}$$

This shows that we can lift any class in $H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right)$ to a class in $H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)$. Hence the inclusion

$$H^1(\mathcal{C}^\bullet)/\left(s_1,\ldots,s_{i+1}\right)H^1(\mathcal{C}^\bullet)\hookrightarrow H^1\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right)$$

is also surjective. This proves that that s_{i+1} is a non-zerodivisor on

$$H^1(\mathcal{C}^\bullet)/(s_1,\ldots,s_i)H^1(\mathcal{C}^\bullet)$$

for $i\in\{0,\ldots,k-1\}$. Therefore, $H^1(\mathcal{C}^\bullet)$ is a Cohen-Macaulay $\mathcal{O}_S$-module. For $dim(L)=2$, we have automatically that

$$H^2\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)\to H^2\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_{i+1}\right)\mathcal{C}^\bullet\right)$$

is surjective. On the other hand, the surjectivity of this map at the H^1-level which we have just proved shows (by using again the connecting homomorphism) that

$$H^2\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)\stackrel{\cdot s_{i+1}}{\longrightarrow}H^2\left(\mathcal{C}^\bullet/\left(s_1,\ldots,s_i\right)\mathcal{C}^\bullet\right)$$

is injective. Then, by the same argument, $H^2(\mathcal{C}^\bullet)$ is Cohen-Macaulay and therefore locally free over $\mathcal{O}_S$. □

Corollary 3.40. *Let $(L,0)\subset(M,0)$ be a complete intersection. Consider a deformation $f:\mathcal{L}\to S$ of L such that the assumptions of the last theorem are fulfilled. Suppose moreover that $(L,0)$ is smoothable and denote by L_ϵ the smooth general fibre of f. Then the following equality holds*

$$dim(H^1(\mathcal{C}^\bullet_{L,0}))=b_1(L_\epsilon)$$

where b_1 denotes the first Betti-number of L_ϵ. For a surface we also get $dim(H^2(\mathcal{C}^\bullet_{L,0}))=b_2(L_\epsilon)$.

Proof. We use the morphism $J:\Omega^1_{\mathcal{L}/S}\to\mathcal{C}^1_{\mathcal{L}/S}$ from page 69. J was seen to be an isomorphism at smooth points of any fibre $\mathcal{L}_s$. Let $D\subset S$ be the discriminant set of f which is a proper subspace by assumption. The last theorem then implies that $f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ is a locally free extension of the cohomology bundle $\bigcup_{\epsilon\in S\setminus D}H^1(\mathcal{L}_\epsilon,\mathbb{C})$ over the discriminant. Moreover, the zero fibre $f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})/\mathbf{m}_{\mathcal{O}_S}f_*\mathcal{H}^1(\mathcal{C}^\bullet_{\mathcal{L}/S})$ is canonically identified with the space $H^1(\mathcal{C}^\bullet_{L,0})$. This proves the first statement. The second one follows by the same argument using the freeness of $f_*\mathcal{H}^2(\mathcal{C}^\bullet_{\mathcal{L}/S})$ for two-dimensional lagrangian singularities. □

This result is already sufficient to calculate the dimension of $\mathcal{H}^i(\mathcal{C}^\bullet_L)$ if L is a product of two curves.

Corollary 3.41. *Let $f \in \mathbb{C}\{x,y\}$ and $g \in \mathbb{C}\{s,t\}$ be two functions defining plane curve singularities $(C,0)$ and $(D,0)$. Then for the lagrangian surface $L := C \times D \subset \mathbb{C}^4$ we have*

$$dim(H^1_{L,0}) = \mu(f) + \mu(g)$$

$$dim(H^2_{L,0}) = \mu(f) \cdot \mu(g)$$

Proof. L is completely integrable, therefore the involutive ideal $\widetilde{\mathcal{I}} = (f+\epsilon_1, g+\epsilon_2)$ ($\epsilon_i \in \mathbb{C}$) is a non-trivial lagrangian deformation. It is obviously a smoothing, so we can apply the last corollary. Then the Künneth formula for the cohomology of a smooth fibre L_ϵ shows that $H^1(L_\epsilon,\mathbb{C}) = H^1(C_\epsilon,\mathbb{C}) \oplus H^1(D_\epsilon,\mathbb{C})$ and $H^2(L_\epsilon,\mathbb{C}) = H^1(C_\epsilon,\mathbb{C}) \otimes H^1(D_\epsilon,\mathbb{C})$. □

3.5 Computations

We are going to use all the techniques developed up to now to calculate the deformation spaces and related invariants for some examples of singular lagrangian varieties. Most of these computations are by large too complicated to be done by hand, but computer algebra turns out to be a quite powerful tool. In particular, we made extensive use of the program *Macaulay2* ([GS]). We will not include the code that has been developed for the calculations in the text, but indicate as explicit as possible how one gets the results. To simplify the calculation, we will only consider the complex case here.

Our main source of examples are lagrangian surface singularities in four space. For surfaces satisfying "Condition P", we have a stratification consisting of three strata: the regular locus L_{reg}, the singular locus (denoted Σ) away from the origin and the origin, which is the unique point with maximal embedding dimension (equal to four). Our aim is to calculate the stalks of the cohomology of $\mathcal{C}^\bullet_L$ at the origin. This will be possible for one important class of examples, these are quasi-homogeneous varieties with positive weights. To be more precise, we suppose that our space L is *strongly quasi-homogeneous* in the sense of [CJNMM96], this means that for each point $p \in L$, we can choose local coordinates of the ambient space such that the defining equations for $(L,p) \subset (M,p)$ become quasi-homogeneous with positive weights. Recall that there is a morphism of DGA's $J : \Omega^\bullet_L \to \mathcal{C}^\bullet_L$ which is an isomorphism on the smooth locus. Moreover, the kernel of this map consists of the torsion subsheaves of $\Omega^\bullet_L$ (see lemma 3.14 on page 70), therefore, there is an injection of complexes $\widetilde{\Omega}^\bullet_L \hookrightarrow \mathcal{C}^\bullet_L$.

Lemma 3.42. *Let $L \subset M$ be strongly quasi-homogeneous. Then*

1. *The de Rham complex $\Omega^\bullet_L$ as well as the complex $\widetilde{\Omega}^\bullet_L$ are resolutions of the constant sheaf $\mathbb{C}_L$.*
2. *Define $\mathcal{E}^\bullet := \mathcal{C}oker\left(\widetilde{\Omega}^\bullet_L \hookrightarrow \mathcal{C}^\bullet_L\right)$. Suppose $dim(L) = 2$. Then $\mathcal{E}^\bullet$ is a two-term complex $\mathcal{E}^1 \xrightarrow{\delta} \mathcal{E}^2$ and we have*

$$\mathcal{H}^1(\mathcal{C}^\bullet_L) \cong \mathcal{K}er\left(\mathcal{E}^1 \xrightarrow{\delta} \mathcal{E}^2\right) \quad \text{and} \quad \mathcal{H}^2(\mathcal{C}^\bullet_L) \cong \mathcal{C}oker\left(\mathcal{E}^1 \xrightarrow{\delta} \mathcal{E}^2\right)$$

Proof. The first statement follows from lemma 1.10 on page 19. For the second one, we first notice that $\mathcal{E}^0 = 0$, Moreover, it follows from lemma 3.15 on page 71 that for surfaces, $\mathcal{E}^p = 0$ for $p > 2$. From the exact sequence

$$0 \longrightarrow (\widetilde{\Omega}^\bullet_L, \delta) \longrightarrow (\mathcal{C}^\bullet_L, \delta) \longrightarrow (\mathcal{E}^\bullet, \delta) \longrightarrow 0$$

we deduce the long exact cohomology sequence

$$\ldots \longrightarrow \mathcal{H}^1(\widetilde{\Omega}^\bullet_L) \longrightarrow \mathcal{H}^1(\mathcal{C}^\bullet_L) \longrightarrow \mathcal{H}^1(\mathcal{E}^\bullet) \longrightarrow$$

$$\mathcal{H}^2(\widetilde{\Omega}^\bullet_L) \longrightarrow \mathcal{H}^2(\mathcal{C}^\bullet_L) \longrightarrow \mathcal{H}^2(\mathcal{E}^\bullet) \longrightarrow \mathcal{H}^3(\widetilde{\Omega}^\bullet_L) \longrightarrow \ldots$$

which gives (due to acyclicity of $\widetilde{\Omega}^\bullet$) $\mathcal{H}^1(\mathcal{C}^\bullet_L) \cong \mathcal{K}er\left(\mathcal{E}^1 \xrightarrow{\delta} \mathcal{E}^2\right)$ and $\mathcal{H}^2(\mathcal{C}^\bullet_L) \cong \mathcal{C}oker\left(\mathcal{E}^1 \xrightarrow{\delta} \mathcal{E}^2\right)$. □

Using this result, we are left with the calculation of the cohomology of the complex $\mathcal{E}^\bullet$. This is still a non-trivial task, as the differential is not $\mathcal{O}_L$-linear. However, this complex is supported on the singular locus which is one-dimensional. This simplifies the calculation considerably.

Let $t \in \mathbf{m}_{\mathcal{O}_L} \subset \mathcal{O}_L$ be any function on L which is finite when restricted to Σ. Let $\widetilde{\Sigma}$ be the normalization of Σ. We choose a coordinate s on the normalization such that in $\mathcal{O}_{\widetilde{\Sigma},0}$ we have $s = t^k$ where k is the degree of the map $t : \Sigma \to \mathbb{C}$.

Lemma 3.43. *The product with $\delta(t)$ induces an $\mathcal{O}_L$-linear morphism $j_t : \mathcal{C}_L^1 \to \mathcal{C}_L^2$ which descends to a morphism on the quotient $j_t : \mathcal{E}^1 \to \mathcal{E}^2$. At points $p \in \Sigma \backslash \{0\}$, this map is an isomorphism.*

Proof. It follows directly from the definition of the product structure that there is a commutative diagram of $\mathcal{O}_L$-linear morphisms

$$\begin{array}{ccc} \Omega_L^1 & \xrightarrow{dt\wedge} & \Omega_L^2 \\ \downarrow J & & \downarrow J \\ \mathcal{C}_L^1 & \xrightarrow{\delta(t)\wedge} & \mathcal{C}_L^2 \end{array}$$

Therefore, we obtain a mapping on the quotient $j_t : \mathcal{E}^1 \to \mathcal{E}^2$ which sends a class $\overline{\phi}$ to $\overline{\delta(t) \wedge \phi}$. To prove the second statement, we have to calculate explicitly the modules $\mathcal{E}_p^1$ and $\mathcal{E}_p^2$ for a decomposable lagrangian germ $(L, p) \cong (L', p) \times (C, 0)$ where $(C, 0)$ is a germ of a smooth curve. We are in the situation of lemma 3.33 on page 85: There is a regular hamiltonian function $h \in \mathcal{O}_{M,p}$ which fibres the germ (L, p) and a regular function $g \in \mathcal{O}_{M,p}$ such that $\{f, g\} = 1$. Then we can choose coordinates (x, y, h, g) of M around p. In these coordinates, the variety L is given by an ideal $I = (f(x, y), h) \subset \mathbb{C}\{x, y, h, g\}$ with symplectic form $dx \wedge dy + dh \wedge dg$. The singular locus L near p is in these coordinates given by the vanishing of x, y and h. Therefore we can assume that on $\widetilde{\Sigma}$, the coordinate s coincides with g around the preimage of p. In particular, g does not vanish around $p \in M$.

Denote the local ring $\mathcal{O}_{L,p}$ by R. The conormal module I/I^2 is a free R-module on the two generators f and h, so that

$$Hom_R(I/I^2, R) = Rn_1 \oplus Rn_2$$

with $n_1(f) = 1, n_2(h) = 0, n_2(f) = 0, n_2(h) = 1$. Obviously we have $Hom_R(\bigwedge^2 I/I^2, R) \cong R(n_1 \wedge n_2)$ and the complex $\mathcal{C}^\bullet_{L,p}$ reads

$$\begin{array}{ccccccccc} 0 & \to & R & \longrightarrow & Rn_1 \oplus Rn_2 & \longrightarrow & R(n_1 \wedge n_2) & \to & 0 \\ & & r & \longmapsto & (\{r,f\},\{r,h\}) & & & & \\ & & & & (a,b) & \longmapsto & \{a,h\} + \{f,b\} & & \end{array}$$

We need to calculate the modules of differential forms on L. We have $\Omega^p_R = \Omega^p_{\mathcal{O}_{M,p}}/(I\Omega^p_{\mathcal{O}_{M,p}} + dI \wedge \Omega^{p-1}_{\mathcal{O}_{M,p}})$. Therefore

$$\begin{aligned} \Omega^1_R &= M_1 \oplus M_2 \\ \Omega^2_R &= M_3 \oplus M_4 \end{aligned}$$

where

$$\begin{aligned} M_1 &:= \frac{R\,dx \oplus R\,dy}{R\,df} \\ M_2 &:= R\,dg \\ M_3 &:= \frac{R\,dx \wedge dy}{R\,df \wedge dx \oplus R\,df \wedge dy} \\ M_4 &:= \frac{R\,dx \wedge dg \oplus R\,dy \wedge dg}{R\,df \wedge dg} \end{aligned}$$

Now the map J can be written down explicitly

$$\begin{array}{rcl} J : M_1 & \longrightarrow & Rn_1 \oplus Rn_2 \\ dx & \longmapsto & (\{x,f\},\{x,h\}) \\ dy & \longmapsto & (\{y,f\},\{y,h\}) \\ J : M_2 & \longrightarrow & Rn_1 \oplus Rn_2 \\ dg & \longmapsto & (\{g,f\},\{g,h\}) = (0,1) \end{array}$$

$$\begin{array}{rcl} J : M_3 & \longrightarrow & R \\ dx \wedge dy & \longmapsto & J(dx) \wedge J(dy) = 0 \\ J : M_4 & \longrightarrow & R \\ dx \wedge dg & \longmapsto & J(dx) \wedge J(dg) \\ dy \wedge dg & \longmapsto & J(dx) \wedge J(dg) \end{array}$$

So $\mathcal{E}^1_p = coker\left(\Omega^1_{L,p} \to \mathcal{C}^1_{L,p}\right)$ is $R\,n_1/\left(RJ(dx) + RJ(dy)\right)$, whereas $\mathcal{E}^2_p = coker\left(\Omega^2_{L,p} \to \mathcal{C}^2_{L,p}\right)$ equals $R\,(n_1 \wedge n_2)/\left(RJ(dx \wedge dg) + RJ(dy \wedge dg)\right)$. If we identify $R\,n_1$ and $R\,(n_1 \wedge n_2)$ with $\Omega_{\mathbb{C}^2,0}/f\Omega_{\mathbb{C}^2,0}$ via the (given) volume form $dx \wedge dy$, then we see that $\mathcal{E}^1_p$ and $\mathcal{E}^2_p$ equals $''H/(f \cdot {''H})$ where $''H$ is the Brieskorn lattice of the function f, see also the discussion before theorem 2.3 on page 53.

Next we calculate the map $j_t : \mathcal{E}^1_p \to \mathcal{E}^2_p$. It follows immediately using the above description of these two modules that

$$\begin{array}{rcl} j_t : \mathcal{E}^1_p & \longrightarrow & \mathcal{E}^1_p \\ a & \longmapsto & a \cdot \{t, h\} \end{array}$$

Moreover, $a \cdot \{t, h\} = a \cdot \{s^k, h\} = a \cdot \{g^k, h\} = a \cdot k \cdot g^{k-1}$. As g does not vanish near p, we see that j_t is an isomorphism between $\mathcal{E}^1_p$ and $\mathcal{E}^2_p$. $\square$

The last lemma shows in particular that $\mathcal{E}^1$ and $\mathcal{E}^2$ are locally free $\mathcal{O}_\Sigma$-modules of rank μ outside of the origin. Here μ is the Milnor number of the transversal curve singularity. We are now able to proceed the calculation of the cohomology of the operator $\delta : \mathcal{E}^1 \to \mathcal{E}^2$. From the fact that the function t is finite on Σ and the last lemma we obtain

Theorem 3.44. *Denote by* $\widetilde{E}^i$ $(i = 1, 2)$ *the germ at zero of the direct image sheaf* $t_*\mathcal{E}^i$. *Denote the induced differential* $t_*\delta : \widetilde{E}^1 \to \widetilde{E}^2$ *again by* δ *and the mapping* $t_*j_t : \widetilde{E}^1 \to \widetilde{E}^2$ *by* i. *The quadruple* $(\widetilde{E}^1, \widetilde{E}^2, i, \delta)$ *defines an* (E, F)*-connection in the sense of [Mal74].*

Proof. We are in the following situation: The modules $\widetilde{E}^i$ are $\mathcal{O}_{\mathbb{C},0}$-modules of rank μ, so it remains only to verify the following relation between t, i and δ:

$$\delta(t \cdot e) = i(e) + t \cdot \delta(e)$$

for any $e \in E^1$. It suffices to do this for the sheaves $\mathcal{E}^i$, that is, we have to show that for any $\Phi \in \mathcal{C}^1_L$ the following relation holds in $\mathcal{C}^2_L$: $\delta(t \cdot \Phi) = j_t(\Phi) + t \cdot \delta(\Phi)$. The function $t \in \mathcal{O}_L$ can be seen as an element in $\mathcal{C}^0_L$, then this relation follows immediately from the fact that $(\mathcal{C}^\bullet, \delta, \wedge)$ is a differential graded algebra. $\square$

To simplify notations, we set $\widetilde{E} = \widetilde{E}^1$ and $\widetilde{F} = \widetilde{E}^2$. To proceed our calculations, we need to work with torsion free modules. This is not really a restriction: The morphisms $\delta, j_t : \widetilde{E} \to \widetilde{F}$ obviously send $Tors(\widetilde{E})$ to $Tors(\widetilde{F})$, so that the cohomology on the torsion part can be calculated explicitly (note that the torsion submodules are artinian). Therefore, we set $E := \widetilde{E}/Tors(\widetilde{E})$ and $F := \widetilde{F}/Tors(\widetilde{F})$ and obtain an (E,F)-connection on the free modules E, F.

We still can not compute the cohomology of δ directly because it is a map of (infinite-dimensional) vector spaces. However, the (E,F) connection defines a meromorphic connection ∇_t on the localization $\mathcal{M} := E \otimes_{\mathbb{C}} \mathbb{C}\{t\}[t^{-1}]$ together with two lattices which are the images of E (resp. F) in $\mathcal{M} = E \otimes_{\mathbb{C}} \mathbb{C}\{t\}[t^{-1}]$ $(= F \otimes_{\mathbb{C}} \mathbb{C}\{t\}[t^{-1}])$. Recall that a lattice is a $\mathbb{C}\{t\}$-submodule of $\mathcal{M}$ of rank (say k) equal to the dimension of $\mathcal{M}$ as $\mathbb{C}\{t\}[t^{-1}]$-vector space. To any such lattice E in $(\mathcal{M}, \nabla_t)$ is associated a set of complex numbers $\alpha_1, \ldots, \alpha_l$ with multiplicities $n_{\alpha_1}, \ldots, n_{\alpha_l}$ such that $\sum_{i=1}^{l} n_{\alpha_i} = k$. This set is called the spectrum of E in $(\mathcal{M}, \nabla_t)$. We recall its definition. Set

$$
\begin{aligned}
C^\alpha &:= \left\{m \in \mathcal{M} \mid \exists N \in \mathbb{N} : (t\nabla_t - \alpha)^N m = 0\right\} \\
\mathcal{V}^{\geq\alpha} &:= \textstyle\bigcup_{\beta \in [\alpha, \alpha+1)} \mathbb{C}\{t\} C^\beta \\
\mathcal{V}^{>\alpha} &:= \textstyle\bigcup_{\beta \in (\alpha, \alpha+1]} \mathbb{C}\{t\} C^\beta
\end{aligned}
$$

The spaces C^α are finite-dimensional $\mathbb{C}$–vector subspaces of $\mathcal{M}$ whereas $\mathcal{V}^{>\alpha}$ and $\mathcal{V}^{>\alpha}$ are $\mathbb{C}\{t\}$-modules of rank k, hence lattices. Any section $m \in \mathcal{M}$ can be decomposed in a series

$$
m = \sum_\alpha s(m, \alpha)
$$

where $s(m, \alpha) \in C^\alpha$. A homogeneous element $s(m, \alpha)$ is also called elementary section. For any $m \in \mathcal{M}$, the non-zero section $s(m, \alpha)$ with minimal α (here one has to choose an order in $\mathbb{C}$ compatible with the usual order in $\mathbb{R}$) is called principal part of m. Then one defines

$$
n_\alpha := dim_{\mathbb{C}} \frac{E \cap V^{\geq\alpha}}{E \cap V^{>\alpha} + tE \cap V^{\geq\alpha}}
$$

Therefore, the spectrum encodes the dimension of the spaces of principal sections of elements from E. The reader might consult [Var83], [Her02] or [Sab02] for further detail. Let us denote the spectrum by $Sp(E,\mathcal{M})$. If $\alpha \in Sp(E,\mathcal{M})$ is a spectral number, then $e^{2\pi i\alpha}$ is an eigenvalue of the monodromy operator $T : H \to H$, where H is the vector space of multivalued sections of $\mathcal{M}$ which are flat with respect to ∇_t. Note that the monodromy does not depend on the lattice, but the spectral numbers do, and this additional information consists in the choice of a logarithm of a given monodromy eigenvalue (the choice of an integer by which the logarithm can be shifted). The following lemma shows how the spectral numbers can be used to calculate the cohomology of the operator δ.

Lemma 3.45. *Let an (E,F,δ,j)-connection be given and set $\mathcal{M} := E \otimes_{\mathbb{C}} \mathbb{C}\{t\}[t^{-1}]$ as above. Denote the image of E in $\mathcal{M}$ again by E. Then we have*

$$ker(E \xrightarrow{\delta} F) \cong \bigoplus_{\alpha \in \mathbb{Z}^{\leq 0}} \frac{E \cap V^{\geq \alpha}}{E \cap V^{>\alpha} + tE \cap V^{\geq \alpha}}$$

Moreover, the dimension of the cokernel is given by the index formula

$$\begin{aligned} dim_{\mathbb{C}}\left(coker(E \xrightarrow{\delta} F)\right) &= dim_{\mathbb{C}}\left(ker(E \xrightarrow{\delta} F)\right) \\ &\quad -rank(E) + dim_{\mathbb{C}}\left(coker(j)\right) \end{aligned}$$

Note that here we suppose that E and F are free, otherwise the dimension of the torsion parts has to be taken into account.

Proof. The "$\supset$" part is clear: Given a principal part e in $C^{-k} \cap E$ for $k \in \mathbb{N}$, one sees immediately that $t^k e$ is annihilated by $t\nabla_t$ and hence by ∇_t as t is invertible on $\mathcal{M}$. Conversely, let e be an element of the kernel of δ, i.e., $\nabla_t e = 0$, then $t\nabla_t e = 0$. Then by choosing a basis of E and decomposing we can suppose that $e = a(t)e_0$ where e_0 is a basis vector. Let $a(t) = t^k \epsilon$, where k is the order of a (so ϵ is a unit). Then

$$0 = (t\nabla_t)(t^k \epsilon e_0) = t^k \left((k\epsilon + t\epsilon')e_0 + t\nabla_t e_0\right)$$

This implies that we obtain a non-zero class in the quotient

$$\left(E \cap V^{\geq -k}\right) / \left(E \cap V^{>-k} + tE \cap V^{\geq -k}\right)$$

which shows the first statement. A proof of the index formula can be found in [Mal74]. □

By this result, we are left with the calculation of the spectral numbers. This is possible due to the following observation, a proof of which can be found in [Her02] (the result is of course much older).

Lemma 3.46. *Let $E \subset (\mathcal{M}, \nabla_t)$ be a logarithmic (or saturated) lattice, i.e., suppose that E is stable under the action of the operator $t\nabla_t$. Then the spectral numbers of $Sp(E, M)$ are the eigenvalues of the residue endomorphism, that is, of the endomorphism*

$$t\nabla_t : E/tE \longrightarrow E/tE$$

This simplifies the whole situation: the residue endomorphism is just a map of finite-dimensional vector spaces, which can easily be calculated. Returning to our situation of the (E, F)-connection coming from the modules $\mathcal{E}^i$ on the singular locus of the lagrangian variety, the problem of calculating the cohomology would be solved if the lattice E were logarithmic. Unfortunately, this is not the case in general, but we can overcome this difficulty using the following trick. Suppose that there is a sublattice $E' \subset E$ such that E' is logarithmic. By the very definition of a lattice, the quotient E/E' is artinian, so one can calculate the cohomology of δ on E/E' explicitly. Using the above lemma, the spectrum $Sp(E', \mathcal{M})$ gives the cohomology of $\delta_{E'}$. It rests to show that in our situation, there is always such a lattice E'.

Lemma 3.47. *Let L be a strongly quasi-homogeneous lagrangian surface singularity. Consider the above defined (E, F)-connection (E, F, δ, j). Then the modules E, F are naturally graded vector spaces and the maps δ and j are homogeneous morphisms such that $deg(t\delta) = deg(j)$. Moreover, there is a submodule $E' \subset E$ with $(t\delta)(E') \subset j(E')$.*

Proof. It is clear that the grading on $\mathcal{O}_M$ induces a grading on $\Omega^\bullet_L$, $\mathcal{C}^\bullet_L$ and thus on the quotient $\mathcal{E}^\bullet$. Note that the exterior differential in the de Rham complex is homogenous of degree zero, but the degree of $\delta^i : \mathcal{C}^i_L \to \mathcal{C}^{i+1}_L$ is $-deg(J)$ where $J : \Omega^1 \to \mathcal{C}^1_L$. Thus also $deg(\delta : \mathcal{E}^1_L \to \mathcal{E}^2_L) = -deg(J)$. If we choose the projection $t \in \mathcal{O}_L$ to be homogeneous, then the $\mathbb{C}\{t\}$-modules E^i are graded and the mappings δ and j are homogenous. It is an easy calculation that $deg(t\delta) = deg(j)$.

We know from lemma 3.43 on page 97 that the cokernel of j is artinian. This implies that there is a certain degree d such that j maps $\oplus_{i \geq d} E^1_i$ isomorphically to $\oplus_{i \geq d} E^2_{i+deg(j)}$. Then $E' := \oplus_{i \geq d} E^1_i$ is the lattice we are looking for. $\square$

The results presented up to this moment implies the following algorithm to calculate the first two cohomologies of the complex $\mathcal{C}^\bullet_{L,0}$ for a quasi-homogeneous surface singularity: The first point is to compute presentations of the modules $\mathcal{C}^i_{L,0}$ and $\Omega^i_{L,0}$ as $\mathcal{O}_{L,0}$-modules as well as the morphisms $J^i : \Omega^i_{L,0} \to \mathcal{C}^i_{L,0}$ for $i = 1, 2$ and the morphism j_t for a convenient function t (which must not vanish on any component of the singular locus of L). The calculation of these modules is standard in computer algebra (see [GP02] or [EGSS02]). On the other hand, computing the morphisms J and j_t involves an implementation of the Poisson-bracket which can of course be done. Nevertheless, J and j_t are $\mathcal{O}_L$-linear thus representable by a matrix. However, this is not true for δ which makes the whole thing complicated.

We obtain presentations for $\mathcal{E}^i$ and j_t (seen as a $\mathcal{O}_{L,0}$-linear map from $\mathcal{E}^1$ to $\mathcal{E}^2$). Now one uses the decomposition in graded parts of $\mathcal{E}^i$ to choose a submodule $\mathcal{E}'$ corresponding to the sublattice E' and calculates the residue endomorphism in any base of E'/tE' as well as the operator δ on the (artinian) modules E/E' and $Tors(\widetilde{E})$. Then the index formula allows us to deduce the dimension of the cokernel of δ, that is, the dimension of the second cohomology of $\mathcal{C}^\bullet_L$.

In the sequel, we will list results for the following examples: the two-dimensional open swallowtail $\Sigma_2 \subset \mathbb{K}^4$, conormal cones of plane curves (these are also surfaces in four space) and some integrable systems in $\mathbb{K}^4$. For the open swallowtail, we obtain.

Theorem 3.48. *The dimensions of the first and second cohomology of* $\mathcal{C}^\bullet_{\Sigma_2,0}$ *are*

$$dim\left(H^1(\mathcal{C}^\bullet_{\Sigma_2,0})\right) = 0 \qquad dim\left(H^2(\mathcal{C}^\bullet_{\Sigma_2,0})\right) = 1$$

Moreover, the spectral numbers for a suitable chosen lattice E' are:

$$Sp(E', \mathcal{M}) = \left\{ \frac{8}{10}, \frac{13}{10}, \frac{22}{10}, \frac{27}{10} \right\}$$

For conormal cones, we present in the following table results for $T^*_C\mathbb{C}^2$ where C is a curve singularity. The exponents added to some values of the spectrum are the multiplicities of that spectral number if different from one. If there are no spectral numbers given, then the modules $\mathcal{E}^i$ are artinian.

C	$dim(H^1)$	$dim(H^2)$	$Sp(E', \mathcal{M})$
$y^2 - x^3$	0	0	
$y^2 - x^5$	0	0	$\frac{4}{5}, \frac{16}{5}$
$y^3 - yx^3$	0	0	9
$y^3 - x^5$	0	0	$\frac{29}{5}, \frac{41}{5}$
$y^3 - x^7$	0	0	$\frac{37}{7}, \frac{61}{7}, \frac{69}{7}, \frac{85}{7}, \frac{93}{7}, \frac{117}{7}$
$y^5 - x^7$	0	0	$\frac{116}{7}, \frac{132}{7}, \frac{148}{7}, \frac{164}{7},$
$y^3 - x^6$	1	1	$\frac{7}{2}, \frac{10}{2}^{(2)}, \frac{13}{2}$
$xy(x+y)(x-y)$	1	1	
$xy(x+y)(x-y)(x+2y)$	2	2	

Finally, we consider integrable systems. We return to the examples in $\mathbb{K}^4$ from table 1.4 on page 36, given by coefficients (λ, μ) and exponents $(\alpha, \beta, \gamma, \delta)$.

λ, μ	$\alpha, \beta, \gamma, \delta$	$dim(H^1)$	$dim(H^2)$	$Sp(E', \mathcal{M})$ (with multiplicity)
$1, 0$	$0, 0, 1, 1$	2	1	$3^{(4)}$
$1, 2$	$0, 2, 1, 0$	3	2	$\frac{2}{2}^{(2)}, \frac{3}{2}^{(2)}, \frac{4}{2}^{(2)}, \frac{5}{2}^{(2)}, \frac{6}{2}^{(2)}$
$1, 3$	$3, 0, 0, 1$	4	3	$\frac{3}{3}^{(2)}, \frac{5}{3}^{(2)}, \frac{7}{3}^{(4)}, \frac{9}{3}^{(4)},$ $\frac{11}{3}^{(4)}, \frac{13}{3}^{(2)}, \frac{15}{3}^{(2)}$
$1, 4$	$4, 0, 0, 1$	5	4	$\frac{4}{4}^{(2)}, \frac{7}{4}^{(2)}, \frac{9}{4}^{(2)}, \frac{10}{4}^{(2)}, \frac{12}{4}^{(2)}, \frac{13}{4}^{(2)},$ $\frac{14}{4}^{(2)}, \frac{15}{4}^{(2)}, \frac{16}{4}^{(2)}, \frac{17}{4}^{(2)}, \frac{18}{4}^{(2)}, \frac{19}{4}^{(2)},$ $\frac{20}{4}^{(2)}, \frac{22}{4}^{(2)}, \frac{23}{4}^{(2)}, \frac{25}{4}^{(2)}, \frac{28}{4}^{(2)}$

In all of the above examples, there is an astonishing symmetry popping up. Comparing the above situation with the classical theory of the spectral numbers of a hypersurface singularity (which is the spectrum of the Brieskorn lattice inside the Gauß-Manin system), one is led to look for a non-degenerate form on the module $(\mathcal{M}, \nabla)$, i.e., a form $(\cdot,\cdot) : \mathcal{M} \otimes \mathcal{M} \to \mathbb{C}\{t\}[t^{-1}]$ such that $d(\cdot,\cdot) = (\nabla\cdot,\cdot) + (\cdot,\nabla\cdot)$. This would imply the symmetry of $Sp(F,\mathcal{M})$ for any lattice $F \subset \mathcal{M}$, particular for the lattice E. Therefore, although the lattice E' we have used to calculate the spectral numbers is not canonically associated to the lagrangian surface $L \subset \mathbb{C}^4$, the observed symmetry is an important hint to the existence of such a duality. One might speculate that it comes (much like in the case of a hypersurface singularity) from the topology of the lagrangian singularity. However, as we are dealing with arbitrary varieties (non-complete intersections which might not even be smoothable), it is much more difficult to use this kind of argument. What we know is that locally around a point $p \in Sing(L)\backslash\{0\}$, L is a product of a curve C_p with a line. Hence one can consider the cohomology $H^1(C_{p_\epsilon}, \mathbb{C})$ of a (canonical) Milnor fibre of such a transversal curve. This is a vector space of a dimension which equals the rank of the modules $\mathcal{E}^i$ at the point p (see the proof of lemma 3.43 on page 97). Speculating further in this direction, we might state the following

Conjecture 3.49. *Let $L \subset \mathbb{C}^4$ a quasi-homogenous lagrangian surface singularity with one-dimensional singular locus, which we denote by Σ. Let μ be the Milnor number of its transversal singularity. Then there is a vector bundle H on $\Sigma^* := \Sigma\backslash 0$ of rank μ such that each fibre is canonically isomorphic to $H^1(C_{p_\epsilon}, \mathbb{C})$. This bundle comes equipped with a flat structure, induced by the symplectic structure of M. Moreover, choosing a projection $t \in \mathcal{O}_L$ as above one obtains a meromorphic bundle H on $\mathbb{C}^*$ and the modules E and F are both locally free extensions over the origin. The constructed connection ∇ on $\mathcal{M}$ (i.e., the connection coming from the morphism δ) coincides with the (conjectured) connection on the topological bundle H. Finally, the Seifert form on the Milnor fibre induces a non-degenerate pairing on $\mathcal{M}$ which explains the symmetry of the spectral numbers.*

We only remark that the main difficulty in proving this speculation is the construction of the topological bundle. It does exist locally around any point p (this is evident due to the product structure), but one needs to construct it without making any choices, only in terms of the ideal I which defines L.

Chapter 4

Isotropic Mappings

This chapter contains mainly calculations of deformation spaces for some simple examples of isotropic mappings. We call any map i from a n-dimensional to a $2n$-dimensional symplectic manifold isotropic iff the pullback $i^*\omega$ of the symplectic form vanishes. Then the image of this map is obviously a lagrangian singularity, but the deformation theory of the map differs considerably from that of its image (which we developed in the last chapter). Unfortunately, there is for the moment no good algorithm (even in the quasi-homogeneous case) which allows one to calculate systematically the deformation spaces. Therefore, we have to restrict ourselves to examples sufficiently simple to be computed by hand. We will mostly be concerned with germs of maps from $\mathbb{K}^2$ to $\mathbb{K}^4$, and we assume them to have rank one. This simplifies the computations.

4.1 Generalities and basic examples

When studying a mapping $f : X \to Y$ between analytic spaces, or even a germ of such a mapping at points $x \in X$ and $f(x) \in Y$, the abstract theory of deformations as developed in the second chapter becomes much more complicated. The main reason is that all objects (modules, complexes and so on) which one has to consider involve two spaces (X and Y) and should therefore "live" on both of them. This idea can indeed be carried out by using the concept of sites and topoi. One can develop a variant of the cotangent complex in this setup. The interested reader might consult [Ill71], [Ill72] or [Buc81]. However, we will consider a much simpler situation, namely a germ of a mapping

$$f : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^m, 0)$$

which we might suppose to be isotropic (in case that $m = 2n$). There are several group actions which one can allow, corresponding to the so-called $\mathcal{R}$, $\mathcal{L}$, $\mathcal{R}-\mathcal{L}$-equivalence etc. We will use $\mathcal{R}-\mathcal{L}$-equivalence. Therefore, the corresponding deformation functor Def_f associates to $S \in$ **Art** an equivalence class of map germs

$$F : (\mathbb{K}^n \times S, 0) \longrightarrow (\mathbb{K}^m \times S, 0)$$

where F_1 and F_2 are isomorphic iff $F_1 = \Phi \circ F_2 \circ \phi$ for analytic isomorphisms $\Phi \in Aut_S(\mathbb{K}^m)$ and $\phi \in Aut_S(\mathbb{K}^n)$. In the symplectic case ($m = 2n$, $\mathbb{K}^m$ symplectic and f isotropic) recall the definition of the functor $IsoDef_f$ (definition 2.6 on page 56): elements of $IsoDef_f(S)$ are isomorphism classes of map germs as above with $(F \circ \pi)^*\omega = 0$ ($\pi : \mathbb{K}^{2n} \times S \to \mathbb{K}^{2n}$ being the projection) with F_1 equivalent to F_2 iff $F_1 = \Phi \circ F_2 \circ \phi$ where $\Phi \in Symp_S(\mathbb{K}^m)$ and $\phi \in Aut_S(\mathbb{K}^n)$. Obviously, Def_f is unobstructed. However, this is not true for $IsoDef_f$ as we will see in the sequel.

The tangent space of Def_f is known to be

$$T^1(f) \cong \frac{f^*\Theta_{\mathbb{K}^m,0}}{df(\Theta_{\mathbb{K}^n,0}) + \Theta_{\mathbb{K}^m,0}}$$

where $df(\Theta_{\mathbb{K}^n,0})$ is the image of the derivative $df : \Theta_{\mathbb{K}^n,0} \to f^*\Theta_{\mathbb{K}^m,0}$ of the map f. It is an important observation that this is not an $\mathcal{O}_{\mathbb{K}^n,0}$-module but only an $\mathcal{O}_{\mathbb{K}^m,0}$-module (because of the term $\Theta_{\mathbb{K}^m,0}$ in the denominator). The structure of the tangent space of $IsoDef_f$ is more subtle. For notational simplicity, we denote $(\mathbb{K}^{2n}, 0)$ by $(M, 0)$ and $(\mathbb{K}^n, 0)$ by $(N, 0)$. Let LV_f be the following vector subspace of $f^*\Theta_{M,0}$:

$$LV_f := \left\{ \sum_{i=1}^{2n} g_i \partial_{x_i} \in f^*\Theta_{M,0} \,|\, (f_i + \epsilon g_i)^*_{i=1,\dots,2n}\omega = 0 \right\}$$

where $x_1, \ldots, x_{2n}$ are coordinates on $(M, 0)$. These are the deformed isotropic mappings. Denote by $\mathcal{H}am_{M,0}$ the sub-vector space of $\Theta_{M,0}$ consisting of germs of hamiltonian vector fields on M. Then $\mathcal{H}am_{M,0}$ lies naturally in LV_f: A deformation of f by an element $X_h \in \mathcal{H}am_{M,0} \subset \Theta_{M,0} \subset f^*\Theta_{M,0}$ is still isotropic, thus an element of LV_f. Moreover, the derivative df maps $\Theta_{N,0}$ into LV_f (this follows directly from the isotropy of f). Then we have

Lemma 4.1. *The tangent space of $IsoDef_f$ is*

$$T^1_{IsoDef}(f) = \frac{LV_f}{df(\Theta_{N,0}) + \mathcal{H}am_{M,0}}$$

Note that this is only a $\mathbb{K}$-vector space.

To illustrate the above facts, we calculate the most basic example, namely, a map germ $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ defining a monomial curve.

Lemma 4.2. *Let $p, q \in \mathbb{N}$, $gcd(p, q) = 1$, $p < q$ and*

$$\begin{array}{rcl} f : (\mathbb{K}, 0) & \longrightarrow & (\mathbb{K}^2, 0) \\ t & \longmapsto & (t^p, t^q) \end{array}$$

be an irreducible germ of a monomial curve singularity. Then we have

$$T^1(f) \cong T^1_{IsoDef}(f) \cong \mathbb{C}^\delta$$

with $\delta = (p-1)(q-1)/2$.

Proof. This can essentially be shown by a close look at a monomial diagram. We will first recall an elementary proof of the equality

$$\dim T^1_{IsoDef}(f) = \delta$$

The following facts will be used: let n be a natural number greater or equal to $(p-1)(q-1)$, then there exists $r, s \in \mathbb{N} := \{0, 1, 2, \ldots\}$ such that $rp + sq = n$. Moreover, in the interval $[0, (p-1)(q-1)-1]$ there are exactly $(p-1)(q-1)/2$ numbers admitting such an representation and they are distributed in the following way: if $n \in [0, \ldots, (p-1)(q-1)-1]$ and $n = rp + sq$ for some $r, s \in \mathbb{N}$, then the number $n' := (p-1)(q-1) - n$

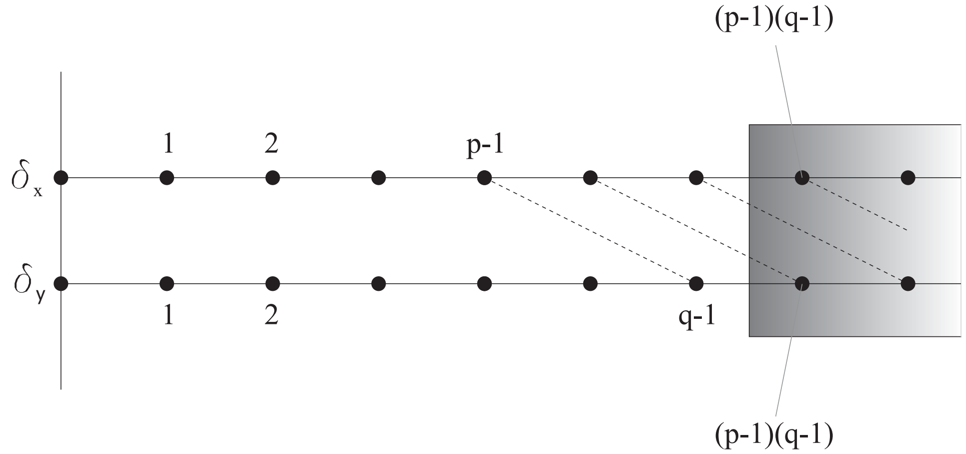


Figure 4.1: Monomial diagram

(this is n "reflected" at $(p-1)(q-1)/2$) can not be represented as $r'p+s'q$ for $r', s' \in \mathbb{N}$. Choosing coordinates x, y in $\mathbb{K}^2$ we have

$$T^1_{Def}(f) = \frac{\mathbb{K}\{t\}\partial_x \oplus \mathbb{K}\{t\}\partial_y}{\mathbb{K}\{t\}(pt^{p-1}, qt^{q-1}) + \mathbb{K}\{t^p, t^q\}\partial_x + \mathbb{K}\{t^p, t^q\}\partial_y}$$

It follows that a deformation of type $t^{rp+sq}\partial_x$ or $t^{rp+sq}\partial_y$ is trivial because the function t^{rp+sq} is in $\mathbb{K}\{t^p, t^q\}$. So a non-trivial deformation consists of terms $t^k\partial_x$ or $t^k\partial_y$ such that k is not representable as $k = rp+sq$. These are *a priori* $2\frac{(p-1)(q-1)}{2} = (p-1)(q-1)$ deformations. But the submodule $\mathbb{K}\{t\}(pt^{p-1}, qt^{q-1})$ causes further identifications: a term $t^k\partial_x$ is equivalent to $\frac{q}{p}t^{k+q-p}\partial_y$ whenever $k \geq p-1$. So in order to count deformations properly we proceed as follows (see figure 4.1): We first take all monomials from the lower row which are not trivial (i.e., not representable as t^{rp+sq}), the we add those from the upper row not related to any of the lower row (those of the form $t^k\partial_x$ with $0 < k < p-1$, these are nontrivial because $k < p < q$). These are $(p-1)(q-1)/2+(p-2)$ deformations not related by an isomorphism. But in the first group (those from the second row) we have some monomials isomorphic to a trivial deformation of type $t^l\partial_x$. A proper count shows that this are exactly $p-2$ ones. So the result is

$$dim\ T^1(f) = \frac{(p-1)(q-1)}{2} = \delta$$

To prove the desired formula for lagrangian deformations, we assume the symplectic form to be $\omega = dx \wedge dy$. Any deformation of f is automatically isotropic, so $LV_f = f^*\Theta_{M,0}$. This leads to:

$$T^1_{IsoDef}(f) = \frac{\mathbb{K}\{t\}\partial_x \oplus \mathbb{K}\{t\}\partial_y}{\mathbb{K}\{t\}(pt^{p-1}, qt^{q-1}) + \{(-\partial_y h \circ f, \partial_x h \circ f) \,|\, h \in \mathcal{O}_{\mathbb{K}^2,0}\}} \tag{4.1}$$

It therefore suffices to prove the following: Let $t^{rp+sq}\partial_x$ or $t^{rp+sq}\partial_y$ be a deformation with $r, s \in \mathbb{N}$. Then it is trivial not only as an ordinary but also as an lagrangian deformation. Let's treat the case $t^{rp+sq}\partial_x$, the other one is similar: We have the following equalities

$$t^{rp+sq}\partial_x = x^r y^s \partial_x = -\partial_y(-\frac{1}{s+1}x^r y^{s+1})\partial_x$$

By the second relation in the denominator of formula 4.1, the last term is equivalent to

$$\begin{aligned} \partial_x(-\frac{1}{s+1}x^r y^{s+1})\partial_y &= -\frac{r}{s+1}x^{r-1}y^{s+1}\partial_y = \\ -\frac{r}{s+1}t^{rp+sq-p+q}\partial_y &= -\frac{r}{s+1}t^{(r-1)p+(s+1)q}\partial_y \end{aligned}$$

On the other hand, it follows from the first relation in equation 4.1 that that

$$\begin{aligned} -\frac{r}{s+1}t^{(r-1)p+(s+1)q}\partial_y &= -\frac{r}{q(s+1)}t^{(r-1)p+sq+1}qt^{q-1}\partial_y \cong \\ -\frac{pr}{q(s+1)}t^{(r-1)p+sq+1}t^{p-1}\partial_x &= -\frac{r}{q(s+1)}t^{rp+sq}\partial_x \end{aligned}$$

This is a contradiction, we get that the deformations

$$-\frac{r}{q(s+1)}t^{rp+sq}\partial_x \quad \text{and} \quad t^{rp+sq}\partial_x$$

are equivalent, which is impossible. So they are zero. □

For lagrangian deformations of a curve $(C, 0)$ (deformation of the image of an isotropic mapping $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$) we had (see formula 2.1 on page 53) that

$$\dim\left(T^1_{LagDef}(C, 0)\right) = \mu > \tau = \dim\left(T^1_{Def}(C, 0)\right)$$

These numbers coincide in the quasi-homogeneous case, so the result on curves of type $t \mapsto (t^p, t^q)$ is not too surprising. However, if the image curve C is not quasi-homogenoues, then the cohomology $H^1(\widetilde{\Omega}_{C,0})$ is not zero. Therefore, we can consider the lagrangian family $(C \times S, 0) \subset ((M, 0), \omega_S)$ where ω_S is a non-trivial deformation of the symplectic form. Equivalently, there is an analytically trivial family $(C_S, 0) \subset (M \times S, 0)$ which is not trivializable by a symplectic automorphism. As the family C_S is trivial in the analytic category, it must be a δ-constant deformation. Therefore, it can be realized by a deformation of the normalization (the isotropic mapping) f, which is also trivial for the functor Def_f but not for $IsoDef_f$. However, as for lagrangian subvarieties, the calculation of the deformation spaces for non-quasihomogenous examples is rather difficult.

The next example we are discussing are mappings having a decomposable lagrangian space as its image. Here we will see that there is no rigidity principle as in the case of deformations of the image: Therefore, we expect $T^1_{IsoDef}(f)$ to be finite only when $T^1_{Def}(f)$ is finite. We use the following notations: Let $M := \mathbb{K}^{2n+2}$ with coordinates $(p_0, q_0, p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic form $\omega = \sum_{i=0}^n dp_i \wedge dq_i$, write M' for the symplectic reduction of M with respect to p_0. Denote by N the space $\mathbb{K}^{n+1}$ with coordinates $x_1, \ldots, x_n, t$ and by N' the space $\mathbb{K}^n$ with coordinates $x_1, \ldots, x_n$.

Theorem 4.3. *Consider the maps*

$$\begin{array}{rcl} f : (N, 0) & \longrightarrow & M \\ (x_1, \ldots, x_n, t) & \longmapsto & (0, t, f_1, g_1, \ldots, f_n, g_n) \end{array}$$

with $f_i \in \mathcal{O}_{N',0}$ *and*

$$\begin{array}{rcl} f' : (N', 0) & \longrightarrow & M \\ (x_1, \ldots, x_n) & \longmapsto & (f_1, g_1, \ldots, f_n, g_n) \end{array}$$

Suppose f' *to be isotropic, i.e.* $f'^*\omega' = 0$ *which implies* $f^*\omega = 0$. *Then we have*

$$T^1_{IsoDef}(f) \cong T^1_{IsoDef}(g) \otimes \mathbb{K}\{t\}$$

Proof. The elements of $LV_f \subset f^*\Theta_{M,0}$ are vector fields of type

$$r_0\partial_{p_0} + s_0\partial_{q_0} + r_1\partial_{p_1} + s_1\partial_{q_1} \ldots + r_n\partial_{p_n} + s_n\partial_{q_n}$$

with $r_i, s_i \in \mathcal{O}_{\mathbb{K}^n,0}$. These coefficients satisfy a certain system of differential equations which is given by the vanishing of the following two-form on $\mathbb{K}^n$:

$$dr_0 \wedge dt + \sum_{i=1}^{n} (df_i \wedge ds_i + dg_i \wedge dr_i)$$

We calculate in the quotient $T^1_{IsoDef}(f)$, thus we can assume s_0 to be zero, as there is a term of type $\mathbb{K}\{x_1, \ldots, x_n, t\}\partial_{q_0}$ in the denominator.

The lagrangian condition can be restated as

$$d_x r_0 \wedge dt = \sum_{i=1}^{n} (d_x s_i + \partial_t s_i dt) \wedge df_i + dg_i \wedge (d_x r_i + \partial_t r_i dt)$$

where d_x denotes the differential with respect to x. This equals the two conditions:

$$\textstyle\sum_{i=1}^{n} (d_x s_i \wedge df_i - dg_i \wedge d_x r_i) = 0$$

$$\textstyle\sum_{i=1}^{n} (\partial_t s_i dt \wedge df_i + dg_i \wedge \partial_t r_i dt) = d_x r_0 \wedge dt$$

Now if $(\mathbf{r}, \mathbf{s}) := (r_1, s_1, \ldots, r_n, s_n)$ is in $T^1_{IsoDef}(f') \otimes \mathbb{K}\{t\}$, then it can be decomposed into $(\mathbf{r}, \mathbf{s}) = \sum_{j=0}^{\infty} (\mathbf{r}, \mathbf{s})_j t^j$ with $(\mathbf{r}, \mathbf{s})_j \in T^1_{IsoDef}(g)$. Then the first condition is obviously satisfied: it is just the fact that $(\mathbf{r}, \mathbf{s})_j$ defines a lagrangian deformation of f for each j. The second equality can be written as

$$\sum_{i=1}^{n} (\partial_t r_i dg_i - \partial_t s_i df_i) = d_x r_0$$

so by the Poincaré lemma, applied to the differential d_x, we must have

$$d_x \sum_{i=1}^{n} (\partial_t r_i dg_i - \partial_t s_i df_i) = 0$$

in order to get a solution. But this means

$$\partial_t \sum_{i=1}^{n} (d_x r_i \wedge dg_i - d_x s_i \wedge df_i) = 0$$

which is just the derivative of the first condition and therefore automatically satisfied. Summarizing, we can assume that $T^1_{IsoDef}(f)$ is given as

$$T^1_{IsoDef}(f) = \frac{r_1\partial_{p_1} \oplus \ldots \oplus r_n\partial_{p_n} \oplus s_1\partial_{q_1} \oplus \ldots \oplus s_n\partial_{q_n}}{\sum_{i=1}^n \partial_{x_i}(f_1, g_1, \ldots, f_n, g_n) + \left\{X_H \mid H \in \mathcal{O}_{\mathbb{K}^{2n+2},0}\right\}}$$

where X_H is the Hamilton vector field associated to H. As we have already seen, each representative $(\mathbf{r}, \mathbf{s})$ of a class in this quotient may by decomposed into a series $(\mathbf{r}, \mathbf{s}) = \sum_{j=0}^{\infty}(\mathbf{r}, \mathbf{s})^{(j)}t^j$, where $(\mathbf{r}, \mathbf{s})^{(j)}$ is a lagrangian deformation of f'. It remains to show that $(\mathbf{r}, \mathbf{s})$ is trivial iff each $(\mathbf{r}, \mathbf{s})^{(j)}$ is a trivial deformation of f'. But this is clear, because the first terms in the denominator do not contain the variable t and in the second one (the Hamilton field, which may contain p_0 and q_0) we do not derive with respect to p_0 or q_0, so the whole denominator may be decomposed as a series in t, too. □

4.2 Corank 1 mappings

In this section we focus on isotropic mappings which are of corank one, that is, map germs from $(\mathbb{K}^n, 0)$ to $(\mathbb{K}^{2n}, 0)$ such that the differential has rank $n-1$ at the origin. The particular case $n = 2$ has been studied in [Giv86] where it is proved that open Whitney umbrellas form an open subset of the space of all isotropic mappings from $\mathbb{R}^2$ to $\mathbb{R}^4$ (in the C^∞-topology). Givental also conjectured that this subset is dense. This has been proved in [Ish92]. We also discuss the smoothness of the functor $IsoDef_f$ in case that f is of corank one and for some other examples.

Let us start be recalling (see theorem 1.18 on page 33) that the open Whitney umbrella $\mathcal{W}_2$ in $\mathbb{K}^4$ is the image of the mapping

$$\begin{array}{rcl} n : \mathbb{K}^2 & \longrightarrow & \mathbb{K}^4 \\ (s, t) & \longmapsto & (-3st, 2t, s^2, s^3) \end{array}$$

Equations for the image have been given in chapter one. In the following paragraphs, we check that n is indeed a stable map, at least in the formal sense (this is of course well known).

Lemma 4.4. *We have $T^1_{IsoDef}(n) = 0$.*

Proof. This calculation will serve as a model for further computations of isotropic mappings from a plane into the four-space. In contrast to the case of curves, we have to take into account the isotropy condition. However, the fact that n is of corank one makes it easy to fulfill this condition. More precisely, any infinitesimal deformation $\widetilde{n}$ is given as $(s,t) \mapsto (-3st + \epsilon a, 2t + \epsilon b, s^2 + \epsilon c, s^3 + \epsilon e)$. with $a, b, c, e \in \mathbb{K}\{s,t\}$ (we avoid the use of the letter d which denotes the exterior differential). The deformed map $\widetilde{n}$ is isotropic iff $d(-3st + \epsilon a) \wedge d(s^2 + \epsilon c) + (2t + \epsilon b) \wedge (s^3 + \epsilon e) = 0$. From $n^*\omega$ and $\epsilon^2 = 0$ we get that this is equivalent to

$$-3d(st) \wedge dc + da \wedge d(s^2) + 2dt \wedge de + db \wedge d(s^3) = 0$$

Therefore the space LV_n is given by all quadruples (a, b, c, e) satisfying this condition. However, we are only interested in $T^1_{IsoDef}(n)$, which is a quotient of LV_n. We have

$$T^1_{IsoDef}(n) = \frac{LV_n}{r_2(-3t, 0, 2s, 3s^2) + r_2(-3s, 2, 0, 0) + X_h \circ n}$$

with r_1 and r_2 arbitrary functions from $\mathbb{K}\{s,t\}$ and X_h the hamiltonian vector field of a function $h \in \mathbb{K}\{x, y, z, w\}$. Therefore, any deformation of type $b\partial_y$ (recall that $LV_n \subset n^*\Theta_{\mathbb{K}^4,0}$) is equivalent to a deformation of type $a\partial_x$. We get the following simplification: Denote by $\widetilde{LV}_n$ the subspace of $\mathbb{K}\{s,t\}\partial_x \oplus \mathbb{K}\{s,t\}\partial_z \oplus \mathbb{K}\{s,t\}\partial_w$ consisting of triples (a, c, e) such that $-3d(st) \wedge dc + da \wedge d(s^2) + 2dt \wedge de = 0$. This is obviously equivalent to

$$\partial_s e = \frac{3}{2}\partial_s c - \frac{3}{2}\partial_t c + s\partial_t a$$

and we have

$$T^1_{IsoDef}(n) = \frac{\widetilde{LV}_n}{r_2(-3t, 0, 2s) + r_2(-3s, 2, 0) + (-\partial_z h, \partial_w h, \partial_x h) \circ n}$$

We see that once we are given a and c, the remaining component e is uniquely determined by the isotropy condition, and for any (a, c) there is (up to constants) a unique e making the deformed map isotropic. Thus it will be sufficient to calculate a vector space basis for the (a, c)-subspace of $\widetilde{LV}_n$ representing the quotient $T^1_{IsoDef}(n)$. A system of generators of this space is given by all monomials $s^k t^l \partial_x$ and $s^k t^l \partial_z$. We have to show that they are all equal to zero in the quotient. We will use the following principle: We take any monomial m and calculate relations (elements of the denominator of the above formula for $T^1_{IsoDef}(n)$) involving m. Here a relation between monomials $m_1 \sim m_2$ means that the difference lies in the denominator. Then it may happen that we get a relation of type $m \sim \lambda m$ where $\lambda \in \mathbb{K}$ is different from one. Thus the difference and therefore also m itself lies in the denominator (as everything is linear over $\mathbb{K}$), i.e., m is zero in $T^1_{IsoDef}(n)$. We start with $m = s^{2k} t^l \partial_x$. We have

$$m = z^k \left(\tfrac{y}{2}\right)^l \partial_x = -\partial_z \left(-\tfrac{1}{k+1} \left(\tfrac{y}{2}\right)^l z^{k+1}\right) \partial_x$$

$$\cong \partial_w \left(-\tfrac{1}{k+1} \left(\tfrac{y}{2}\right)^l z^{k+1}\right) \partial_y + \partial_x \left(-\tfrac{1}{k+1} \left(\tfrac{y}{2}\right)^l z^{k+1}\right) \partial_z = 0$$

Thus the deformation given by m is trivial.

Now let $m := s^{2k} t^l \partial_z$. This is a bit more complicated, but a very typical calculation as we will see later.

$$m = z^k \left(\tfrac{y}{2}\right)^l \partial_z = \partial_x \left(x \left(\tfrac{y}{2}\right)^l z^k\right) \partial_z$$

$$\cong \partial_w \left(x \left(\tfrac{y}{2}\right)^l z^k\right) \partial_y - \partial_z \left(x \left(\tfrac{y}{2}\right)^l z^k\right) \partial_x$$

$$\cong -kz^{k-1} x \left(\tfrac{y}{2}\right)^l \partial_x = -ks^{2k-2}(-3st) t^l \partial_x$$

$$= 3t \left(ks^{2k-1} t^l\right) \partial_x \cong 2ks^{2k} t^l \partial_z$$

We are precisely in the situation described above: as $2k \neq 1$, we conclude that m is zero in the quotient. Lets now $m := s^{2k+1} t^l \partial_z$. Then

$$m = 2s \left(\frac{1}{2} s^{2k} t^l\right) \partial_z \cong \frac{3}{2} s^{2k} t^l \partial_z$$

But this last term was already seen to be zero. The last monomial is of type $m := s^{2k+1}t^l\partial_x$. Here we have

$$m \cong \tfrac{2}{3}s^{2k}t^l\partial_y = \tfrac{2}{3}z^k\left(\tfrac{y}{2}\right)^l\partial_y \cong$$

$$= \tfrac{2}{3}\partial_w\left(wz^k\left(\tfrac{y}{2}\right)^l\right)\partial_y \cong \tfrac{2}{3}\partial_x\left(wz^k\left(\tfrac{y}{2}\right)^l\right)\partial_z - \tfrac{2}{3}\partial_z\left(wz^k\left(\tfrac{y}{2}\right)^l\right)\partial_x$$

$$\cong -\tfrac{2}{3}kwz^{k-1}\left(\tfrac{y}{2}\right)^l\partial_x = -\tfrac{2}{3}ks^3s^{2k-2}t^l\partial_x = -\tfrac{2}{3}ks^{2k+1}t^l\partial_x$$

This shows that also in this case m is a trivial deformation. The proof is finished. □

This calculation also yields the idea of the proof for the following fact.

Theorem 4.5. *Let $i : (\mathbb{K}^2, 0) \to (\mathbb{K}^4, 0)$ the germ of any isotropic mapping of corank one. Then $IsoDef_i$ is smooth.*

Proof. Let i be given as $(s,t) \mapsto (a,b,c,e)$ with $\partial_t e$ non-vanishing at the origin. Then by a coordinate change in $\mathbb{K}^2$ (this does not affect the symplectic form) we can assume that $e = t$. Now it is easy to see that any deformation over $Spec(A_n) = Spec(\mathbb{K}[\epsilon]/\epsilon^{n+1})$ is equivalent to one of the following type

$$\begin{array}{rcl} i_n : (\mathbb{K}^2,0)\times Spec(A_n) & \to & (\mathbb{K}^4,0) \\ (s,t,\epsilon) & \mapsto & (a+\sum\limits_{k=1}^{n}\epsilon^k a_k, b+\sum\limits_{k=1}^{n}\epsilon^k b_k, c+\sum\limits_{k=1}^{n}\epsilon^k c_k, t) \end{array}$$

with $a_k, b_k, c_k \in \mathbb{K}\{s,t\}$. A deformation over $Spec(A_{n+1})$ of type $i_n + (a_{n+1}\epsilon^{n+1}, b_{n+1}\epsilon^{n+1}, c_{n+1}\epsilon^{n+1}, t)$ (one can always reduce to this case as above) is lagrangian iff

$$dt \wedge db_{n+1} = \sum_{k=0}^{n+1} da_i \wedge dc_{n+1-i} + da_{n+1-i} \wedge dc_i$$

where we set $a_0 := a$ and $c_0 := c$. But this is equivalent to

$$(\partial_s b_{n+1})\, dt \wedge ds = \sum_{k=0}^{n+1} da_i \wedge dc_{n+1-i} + da_{n+1-i} \wedge dc_i$$

which can always be satisfied. Therefore, any given deformation over the space $Spec(A_n)$ can be extended over $Spec(A_{n+1})$ which gives the smoothness of $IsoDef_i$ by lemma A.21 on page 155. □

The following example, taken from [Ish96], shows that there are corank two mappings having obstructed deformations.

Theorem 4.6. *Consider the map-germ*

$$\begin{array}{rcl} i : (\mathbb{K}^2, 0) & \longrightarrow & (\mathbb{K}^4, 0) \\ (s,t) & \longmapsto & (s^2, t^2, 0, 0) \end{array}$$

Then $IsoDef_i$ is not smooth.

Proof. We will exhibit an infinitesimal deformation which cannot be extended to higher order. Consider

$$\begin{array}{rcl} i_1 : (\mathbb{K}^2, 0) \times Spec(A_1) & \longrightarrow & (\mathbb{K}^4, 0) \\ (s,t,\epsilon) & \longmapsto & (s^2 + \epsilon t, t^2, \epsilon s, \epsilon t) \end{array}$$

Obviously, we have $i_1^*\omega = 0$, so $i_1 \in LV_i$. It can be easily checked that the class of i_1 in $T^1_{IsoDef}(i)$ is non-zero. Any extension i_2 of i_1 over $Spec(A_2)$ is of the form $(s,t,\epsilon) \mapsto (s^2 + \epsilon t + \epsilon^2 a, t^2 + \epsilon^2 b, \epsilon s + \epsilon^2 c, \epsilon t + \epsilon^2 e)$ with $a, b, c, e \in \mathbb{K}\{s,t\}$. Then

$$i_2^*\omega = d(s^2) \wedge dc + dt \wedge ds + d(t^2) \wedge de = (1 - 2s\partial_t c + 2t\partial_s e) dt \wedge ds$$

This form is non-zero at the origin for any (a,b,c,e) showing that there is no isotropic extension of i_1 over A_2. □

Note, however, that in this example the tangent space $T^1_{IsoDef}(i)$ is not finite-dimensional. In fact, is is not so obvious how to find examples of maps of rank zero with finite dimensional tangent space. As explained before, it is unlikely that maps i where $T^1_{Def}(i)$ is not finite have finite-dimensional tangent space for the functor *LagDef*. So we first have to look for rank zero maps i such that $dim(T^1_{Def}(i)) < \infty$. Consider the following example

$$\begin{array}{rcl} i : (\mathbb{K}^2, 0) & \longrightarrow & (\mathbb{K}^4, 0) \\ (s,t) & \longmapsto & (s^3, t^3, \frac{1}{3}st^3 + \frac{1}{4}t^4, \frac{1}{4}s^4 + \frac{1}{3}s^3 t) \end{array}$$

Using standard methods in computer algebra (e.g., calculation of a presentation of $i^*\Theta_{\mathbb{K}^4,0}$ as a $\mathcal{O}_{L,0}$-module, where L is the image of i), we obtain that

$$dim\left(T^1_{Def}(i)\right) = 234$$

It is of course very hard to detect the dimension of $dim\left(T^1_{IsoDef}(i)\right)$. There should be simpler examples (with smaller codimension), but it is not so clear how to construct them.

4.3 Symplectic and Lagrange stability

In this section we review the results of Givental and Ishikawa concerning the open Whitney umbrella as generic singularity of corank one isotropic maps from a plane into four space. We work only over $\mathbb{R}$ here. The results are valid in the C^∞-category. First we give a slightly different definition of the open Whitney umbrella in *any* dimension. They are given as the images of the following isotropic mappings.

Definition 4.7. *Let $n, k \in \mathbb{N}$ and $k \leq \left[\frac{1}{2}n\right]$. Define the following map*

$$\begin{array}{rcl} f_{n,k} : (\mathbb{R}^n, 0) & \longrightarrow & (\mathbb{R}^{2n}, 0) \\ (x_1, \ldots, x_{n-1}, z) & \longmapsto & (p_1, \ldots, p_n, q_1, \ldots, q_n) \end{array}$$

where

$$q_i := x_i \qquad i = 1, \ldots, n-1$$

$$q_n := \tfrac{z^{k+1}}{(k+1)!} + \textstyle\sum_{j=1}^{k-1} x_j \tfrac{z^{k-j}}{(k-j)!}$$

$$p_n := \textstyle\sum_{j=0}^{k-1} x_{k+j} \tfrac{z^{k-j}}{(k-j)!}$$

$$p_i := \int \left(\tfrac{\partial p_n}{\partial x_i}\tfrac{\partial q_n}{\partial z} - \tfrac{\partial q_n}{\partial x_i}\tfrac{\partial p_n}{\partial z}\right) dz \qquad i = 1, \ldots, n-1$$

Obviously, if we take $k = 0$, we get just a smooth lagrangian (subvector) space in $\mathbb{R}^{2n}$. Moreover, for any n, k, we have that $f_{n,k} = f_{2k,k} \times f_{n-2k,0}$. Therefore, as before the only interesting case is $n = 2k$

and corresponds to the open Whitney umbrella $\mathcal{W}_{2k}$ as introduced in definition 1.18 on page 33. However, by choosing p_i and q_i as coordinates on $\mathbb{R}^{2n}$, we fix an identification of $\mathbb{R}^{2n}$ with $T^*\mathbb{R}^n$ which is not the same as in the definition of the open Whitney whitney umbrella as conormal cone of the open swallowtail. For $n = 2$, it is the cotangent fibration we have used to calculate the front (the composed Whitney umbrella, see figure 1.5 on page 32).

Theorem 4.8. *Let $n = 2k$ and denote by $Iso(\mathbb{R}^n, \mathbb{R}^{2n})$ the space of isotropic mappings from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ of corank one, equipped with the Whitney C^∞-topology. Then there is a dense open set $W \subset Iso(\mathbb{R}^n, \mathbb{R}^{2n})$ with the following property: Let $i \in W$ be given, then for any point $x \in \mathbb{R}^n$ there is a neighborhood U of x in $\mathbb{R}^n$ and a neighborhood V of i in W such that the restriction of all $j \in V$ to $U \subset \mathbb{R}^n$ is symplectically left-right equivalent to $f_{2n,n}$.*

We will give the main ideas of Ishikawa's proof without carrying out all details. The first point is the following equivalence between isotropic map germs and germs of parameterized fronts. First fix an identification of $\mathbb{R}^{2n}$ with $T^*\mathbb{R}^n$, denote the base space by B and by π the projection $T^*B \to B$. Moreover, we abbreviate the source $\mathbb{R}^n$ of the isotropic maps by N. Then for any $\varphi \in Iso(N, T^*B)$ we have the generating function $F \in \mathcal{E}_{N,0}$, i.e., a function such that $de = \varphi^*\alpha$, where α is the Liouville form on T^*B (see the definition on page 20). Set $\psi := \pi \circ \varphi$. We say that two maps $\varphi, \varphi' \in Iso(N, T^*B)$ are *Lagrange* equivalent iff they are symplectically left-right equivalent and if the symplectomorphism respects the bundle structure given by π.

Lemma 4.9. *Two isotropic maps $\varphi, \varphi' \in Iso(N, T^*B)$ are Lagrange equivalent iff there is $\sigma \in Aut(N)$, $\tau \in Aut(B)$, and a function $S \in \mathcal{E}_{B,0}$ such that*

$$\tau \circ \psi' = \psi \circ \sigma \quad \textit{and} \quad F = F' \circ \sigma + S \circ \psi$$

Proof. Suppose first that φ and φ' are equivalent. Then there is an automorphism σ of N and $\Phi \in Symp(T^*B)$ respecting the fibration given by π such that $\varphi' \circ \sigma = \Phi \circ \varphi$ This implies that $\Phi^*\alpha = \alpha + \pi^* d\widetilde{S}$.

Hence,

$$\begin{aligned} d(\sigma^* F') &= \sigma^* \varphi'^* \alpha = \\ \varphi^* \left(\alpha + \pi^* d\widetilde{S}\right) &= d\left(F + \psi^* \widetilde{S}\right) \end{aligned}$$

Therefore, $\sigma^* F' = F + \psi^* \widetilde{S} + c$ for some constant c and by setting $S = c - \widetilde{S}$ we obtain σ, τ and S as required. On the other hand, suppose σ, τ and S be given. Then $\Phi := \pi^* \tau + dS$ is a symplectomorphism respecting the bundle structure and we have $\Phi^* \alpha = \alpha + \pi^* d\widetilde{S}$. It follows that

$$\begin{aligned} (\Phi \circ \varphi \circ \sigma^{-1})^* \alpha &= (\varphi \circ \sigma^{-1})^* (\alpha + \pi^* d\widetilde{S}) = \\ {\sigma^{-1}}^* (dF - d\psi^* S) &= {\sigma^{-1}}^* d(\sigma^* F') = dF' = \varphi'^* \alpha \end{aligned}$$

The composition of both φ' and $\Phi \circ \varphi \circ \sigma^{-1}$ with π equals ψ' and the pullback of the Liouville form by these two maps coincides, as we have just proved. Therefore, we also have $\Phi \circ \varphi \circ \sigma^{-1} = \varphi'$. □

Note that we did not made use of the fact that the maps under consideration are of corank one. In that case, we can say more.

Lemma 4.10. *Write the isotropic map $\varphi : N \to T^* B$ in the form*

$$(x_1, \ldots, x_{n-1}, z) \longmapsto (p_1, \ldots, p_{n-1}, v(\mathbf{x}, z), x_1, \ldots, x_{n-1}, u(\mathbf{x}, z))$$

Then the generating function is

$$F(\mathbf{x}, z) = \int_0^z v(\mathbf{x}, t) \partial_t u(\mathbf{x}, t) dt + b(\mathbf{x})$$

for a function $b \in \mathcal{E}_{\mathbb{R}^{n-1},0}$ (the ring of C^∞-functions in the variables $x_1, \ldots, x_{n-1}$).

Proof. By definition, $dF = \sum_{i=1}^{n-1} p_i dx_i + v du$. On the other hand, $dF = \sum_{i=1}^{n} \partial_{x_i} F \, dx_i + \partial_z F \, dz$, which implies that $\partial_z F = v \partial_z u$. □

The important point is that given functions v and u, one can construct an isotropic mapping of the above type in an essentially unique way.

Lemma 4.11. *Suppose that $v(\mathbf{x},0)=0$. Then the map*

$$\begin{array}{rcl}\varphi_v : N & \longrightarrow & T^*B \\ (x_1,\dots,x_{n-1},z) & \longmapsto & (p_1,\dots,p_{n-1},v,x_1,\dots,x_{n-1},u)\end{array}$$

where

$$p_i := \int_0^z (\partial_{x_i} v \cdot \partial_t u - \partial_{x_i} u \cdot \partial_t v)\, dt \quad i \in \{1,\dots,n-1\}$$

is isotropic. Moreover, let $\varphi' : N \to T^*B$ be isotropic such that $\pi\circ\varphi' = \pi\circ\varphi$. Then φ' is Lagrange equivalent to the map $\varphi_{v'}$ where $v'(\mathbf{x},z) := (p_n\circ\varphi')(\mathbf{x},z) - (p_n\circ\varphi')(\mathbf{x},0)$.

This lemma is proved by comparing the generating functions and applying lemma 4.9 on page 120. In particular, by taking $u := \frac{z^{k+1}}{(k+1)!} + \sum_{j=1}^{k-1} x_j \frac{z^{k-j}}{(k-j)!}$ and $v := \sum_{j=0}^{k-1} x_{k+j} \frac{z^{k-j}}{(k-j)!}$, one obtains that $f_v \cong f_{n,k}$.

Now the proof of the theorem goes as follows: First one has to detect the open dense subset $W \subset Iso(N,T^*B)$ all germs of which are equivalent to the open Whitney umbrella. This set is determined by the following condition: a map germ $\varphi = (p_1,\dots,p_{n-1},v,\psi)$ is in W iff the map $\widetilde{\psi} = (\psi, v) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a Morin singularity (see [Mor65]), i.e., if the r-jet $j^r\widetilde{\psi}$ is transverse to the *Thom-Boardman-symbol* $\Sigma^{1_k,0}$ inside the r-jet space $J^r(V,\mathbb{R}^{n+1})$ where $r = \left[\frac{n}{2}\right]+2$ and $k \in \{0,\dots,r\}$ (see, e.g., [GG80] for definitions). It follows from the last lemma that the set W defined in this way is open, because $Iso(N,T^*B)$ carries the topology induced from $C^\infty(N,T^*B)$.

Given a map $\varphi \in W$, it is not difficult to see that it is symplectically equivalent to $\widetilde{\varphi}(\mathbf{x},z) = (p_1,\dots,p_{n-1},v,q_1,\dots,q_{n-1},u)$ with

$$\begin{array}{rcl} q_i & = & x_i \quad \forall\, i \in \{1,\dots,n-1\} \\ q_n & =: & u(\mathbf{x},z) = \frac{z^{k+1}}{(k+1)!} + \sum_{j=1}^{k-1} x_j \frac{z^{k-j}}{(k-j)!} \\ p_n & =: & v(\mathbf{x},z) \end{array}$$

for some v with the property that $\frac{\partial^l v}{\partial z^l}_{|(0,0)} = 0$ for $l \in \{0,\dots,k\}$. However, to do this transformation, it is sometimes necessary to interchange the coordinates p and q, therefore here we only have symplectic but not Lagrange equivalence.

It follows that for the generating function F of φ we have $\partial_z F = v \cdot \partial_z u$. Now the main point in the proof is to consider the algebra

$$H_\psi = \{e \in \mathcal{E}_{N,0} \,|\, \exists \varphi : de = \varphi^* \alpha \ , \ \ \pi \circ \varphi = \psi\}$$

of all generating functions of isotropic mappings lifting a given map $\psi : N \to B$. One can show that H_ψ is naturally a $\mathcal{E}_{B,0}$-module via ψ and that it is generated by functions $1, H_1, \ldots, H_k$ with

$$H_l := \int_0^z \frac{t^l}{l!} \left(\frac{t^k}{k!} + \sum_{j=1}^k a_j(\mathbf{x}) \frac{t^{k-j}}{(k-j)!} \right) dt$$

for *fixed* functions $a_i \in \mathbf{m}_{\mathbb{R}\{\mathbf{x}\}}$. At this point it is necessary to use Malgrange's preparation theorem for differential algebras. We obtain that $F = b_0 \circ \psi + \sum_{j=1}^k b_j \circ \psi H_j$ where $b_i : B \to \mathbb{R}$. This implies (using the chain rule)

$$v = (\partial_{q_n} b_0) \circ \psi + \sum_{j=1}^n (\partial_{q_n} b_j) \circ \psi H_j + \sum_{j=1}^k b_j \circ \psi \frac{z^j}{j!}$$

Now it is possible to show that the map

$$\begin{array}{rcl} \sigma : N & \longrightarrow & N \\ (\mathbf{x}, z) & \longrightarrow & \left\{ \begin{array}{ll} x_i & (1 \le i \le k-1, 2k \le i \le n) \\ b_{2k-i} \circ \psi & (k \le i \le 2k-1) \end{array} \right. \end{array}$$

is an automorphism of N leaving H_j $(j \in \{0, \ldots, k\})$ invariant and that there is an automorphism τ of B such that $\psi \circ \sigma = \tau \circ \psi$. Moreover, the generating function F satisfies

$$F = b_0 \circ \psi + \sum_{j=1}^k b_j \circ \psi H_j = b_0 \circ \psi + F' \circ \sigma$$

for a function

$$F' = \sum_{j=1}^k x_{2k-j} H_j = \int_0^z \left(\sum_{j=1}^k x_{2k-j} \frac{t^j}{j!} \right) \partial_u dt$$

This is the generating function of the open Whitney umbrella $f_{n,k} = f_{2k,k}$ which proves the theorem by applying lemma 4.9 on page 120.

We obtain as an immediate consequence.

Corollary 4.12. *The only stable isotropic map germ from* $\mathbb{R}^n$ *to* $\mathbb{R}^{2n}$ *of corank one is the open Whitney umbrella.*

We finish this section by remarking that the subsequent papers of Ishikawa (see in particular [Ish96]) contains also a treatment of the above questions with respect to the Lagrange automorphism group, that is, the semi-direct product of $Aut(N)$ with the subgroup of $Symp(T^*B)$ consisting of symplectomorphisms preserving the Lagrange fibration $\pi : T^*B \to B$.

4.4 Further computations and conjectures

In this section we study isotropic mappings of corank one which are not symplectically equivalent to open Whitney umbrellas. We calculate several invariants attached to them, the most difficult one being its lagrangian codimension, that is, the dimension of T^1_{IsoDef}. It seems that there is always a linear relation between this dimension and some other invariants. More precisely, we will compare the dimension of $T^1_{IsoDef}(\varphi)$ for a mapping $\varphi : (\mathbb{K}^2, 0) \to (\mathbb{K}^4, 0)$ with the dimension of the usual $T^1_{Def}(\varphi)$ as well as with two other invariants: namely, the dimension of the module of relative differential forms with respect to the mappings φ, i.e., $\Omega_{\mathbb{K}^2,0}/\varphi^*\Omega^2_{\mathbb{K}^4,0}$ and with the δ-invariant. Recall that this is the dimension of the quotient $\varphi_*\mathcal{O}_{\mathbb{K}^2,0}/\mathcal{O}_{L,0}$ where $L := Im(\varphi)$. The modules $\Omega_{\mathbb{K}^2,0}/\varphi^*\Omega^2_{\mathbb{K}^4,0}$ and $\varphi_*\mathcal{O}_{\mathbb{K}^2,0}/\mathcal{O}_{L,0}$ are supported on the critical locus of the map φ (resp. on its image), therefore, they will be in general artinian only if the critical locus is isolated.

To obtain other examples than open Whitney umbrellas, we use theorem 4.3 on page 112: Any decomposable isotropic mapping deforms into infinitely many corank one maps. Take the A_{2k+1}-singularity, crossed with a line

$$\begin{array}{rcl}\varphi : (\mathbb{K}^2, 0) & \longrightarrow & (\mathbb{K}^4, 0) \\ (s, t) & \longmapsto & (s^2, t, s^{2k+1}, 0)\end{array}$$

According to lemma 4.2 on page 109 and theorem 4.5 on page 117, the lagrangian deformations are of the following form

$$\widetilde{\varphi} : (s,t) \longmapsto \Big(s^2, t, s^{2k+1} + \sum_{i=1}^{2k} s^{2k-i}\epsilon_i(t),$$

$$\sum_{i=1}^{2k} \tfrac{2}{2(k+1)-i} s^{2(k+1)-i}\epsilon_i'(t) \Big)$$

Similar formulas can be written down for deformations of more general mappings of type $(s,t) \mapsto (s^p, t, s^q, 0)$. Of all deformations obtained in this way we are mostly interested in those which are quasi-homogeneous in the variables (s,t). We will consider the following examples of map germs $\varphi_i : (\mathbb{K}^2, 0 \to (\mathbb{K}^4, 0)$

$$\begin{array}{lll}
\varphi_1 : (s,t) & \longmapsto & \big(s^2, t, s^3 + st^2, \frac{4}{3}s^3t\big) \\
\varphi_2 : (s,t) & \longmapsto & \big(s^2, t, s^5 + st^4, \frac{8}{3}s^3t^3\big) \\
\varphi_3 : (s,t) & \longmapsto & \big(s^2, t, s^7 + st^6, 4s^3t^5\big) \\
\varphi_4 : (s,t) & \longmapsto & \big(s^2, t, s^9 + st^8, \frac{16}{3}s^3t^7\big) \\
\varphi_5 : (s,t) & \longmapsto & \big(s^3, t, s^5 + st^4, 3s^4t^3\big)
\end{array}$$

The calculation of the dimensions of $T^1_{Def}(\varphi)$, $\Omega_{\mathbb{K}^2,0/\mathbb{K}^4,0}$ and of the quotient $\varphi_*\mathcal{O}_{\mathbb{K}^2,0}/\mathcal{O}_{L,0}$ are standard due to the fact that all objects involved here are modules over either $\mathcal{O}_{\mathbb{K}^2,0}$, $\mathcal{O}_{\mathbb{K}^4,0}$ or $\mathcal{O}_{L,0}$. One obtains the following results (where we denote by l the length of a module and by $t^1_{Def}(\varphi)$ the number $l(T^1_{Def}(\varphi)))$.

φ	$t^1_{Def}(\varphi))$	$l\left(\Omega_{\mathbb{K}^2,0/\mathbb{K}^4,0}\right)$	$l\left(\frac{\varphi_*\mathcal{O}_{\mathbb{K}^2,0}}{\mathcal{O}_{L,0}}\right)$
$\big(s^2, t, s^3 + st^2, \frac{4}{3}s^3t\big)$	3	5	2
$\big(s^2, t, s^5 + st^4, \frac{8}{3}s^3t^3\big)$	10	19	4
$\big(s^2, t, s^7 + st^6, 4s^3t^5\big)$	21	55	6
$\big(s^2, t, s^9 + st^8, \frac{16}{3}s^3t^7\big)$	36	97	8
$\big(s^3, t, s^5 + st^4, 3s^4t^3\big)$	28	77	8

As already said, the computation of the dimension of $T^1_{IsoDef}(\varphi)$ is much more involved. However, for the first four examples one has the advantage that both t and s^2 are elements of the ring $\mathcal{O}_{L,0}$ which simplifies everything. We have already seen an example of the calculation for a map with this property (the open Whitney umbrella), so we will not reproduce the details here. However, for φ_5 things are more complicated. Here we have only $t, s^3 \in \mathcal{O}_{L,0}$. Although the computations are in principle the same as before, one has to be much more carefully. Therefore, we will give prove the following theorem in some detail for the map φ_5.

Theorem 4.13. *The lagrangian codimension of the above maps is as follows*

$$dim_{\mathbb{K}}(T^1_{IsoDef}(\varphi_1)) = 1 \quad ; \quad dim_{\mathbb{K}}(T^1_{IsoDef}(\varphi_2)) = 6 \ ;$$

$$dim_{\mathbb{K}}(T^1_{IsoDef}(\varphi_3)) = 15 \quad ; \quad dim_{\mathbb{K}}(T^1_{IsoDef}(\varphi_4)) = 28 \ ;$$

$$dim_{\mathbb{K}}(T^1_{IsoDef}(\varphi_5)) = 20$$

The proof of the last equality will be given in several steps. By definition, we have to compute the dimension of

$$T^1_{IsoDef}(\varphi_5) \quad := \quad \frac{(a,b,c,e) \in \mathcal{O}^4_{\mathbb{K}^2,0} \,|\, \partial_s e = 3s^2\partial_t c + t^4\partial_s a - 4st^3\partial_t a}{d\varphi_5(\Theta_{\mathbb{K}^2,0}) + \mathcal{H}am_{\mathbb{K}^4,0}}$$

where $\mathcal{H}am_{\mathbb{K}^4,0}$ denotes space of Hamilton vector fields on $(\mathbb{K}^4, 0)$. Substituting the map we obtain

$$T^1_{IsoDef}(\varphi_5) =$$

$$\Big((a,b,c,e) \in \mathbb{K}\{s,t\}^4 \,|\, \partial_s e = 3s^2\partial_t c + t^4\partial_s a - 4st^3\partial_t a\Big)\Big/$$

$$\Big(\mathbb{K}\{s,t\}(3s^2, 0, 5s^4 + t^4, 12s^3t^3) + \mathbb{K}\{s,t\}(0,1,t^4,9s^4t^2) +$$
$$\{\varphi_5^{-1}(-\partial_z h, -\partial_w h, \partial_x h, \partial_y h) \mid h \in \mathcal{O}_{\mathbb{K}^4,0}\}\Big)$$

$$\cong \frac{(a,b,c) \in \mathbb{K}\{s,t\}^3}{\mathbb{K}\{s,t\}(3s^2,0,5s^4+t^4) + \mathbb{K}\{s,t\}(0,1,4st^3) + \varphi_5^{-1}(-\partial_z h, -\partial_w h, \partial_x h)}$$

Now we have to analyze this quotient "by hand". We consider only monomial deformations, as everything is $\mathbb{K}$-linear. The first case is:

$$s^{3p}t^l\partial_x = x^py^l\partial_x = -\partial_z\left(-zx^py^l\right)\partial_x \cong -\partial_x\left(-zx^py^l\right)\partial_z$$

$$= pzx^{p-1}y^l = p(s^5+st^4)s^{3p-3}t^l\partial_z = p\left(s^{3p+2}t^l + s^{3p-2}t^{l+4}\right)\partial_z$$

On the other hand, we have for $s > 0$

$$s^{3p}t^l\partial_x = 3s^2\left(\tfrac{1}{3}s^{3p-2}t^l\right)\partial_x \cong (5s^4+t^4)\left(\tfrac{1}{3}s^{3p-2}t^l\right)\partial_z$$

$$= \left(\tfrac{5}{3}s^{3p+2}t^l + \tfrac{1}{3}s^{3p-2}t^{l+4}\right)\partial_z$$

This is obviously impossible except for the zero coefficient. Note that the restriction $s > 0$ is not a real one, as the monomial $t^l\partial_x$ is easily seen to be a trivial deformation (can be trivialized by the hamiltonian field X_{-zy^l}). So we have:

Lemma 4.14. *The deformations $s^{3p}t^l\partial_x$ are trivial for all $p, l \in \mathbb{N}$. The same argument works in the case $s^{3p}t^l\partial_z$.*

Let us analyze the more complicated cases.

$$s^{3p+1}t^l\partial_x = 3s^4t^3\left(\tfrac{1}{3}s^{3(p-1)}t^{l-3}\right)\partial_x = \tfrac{1}{3}wx^{p-1}y^{l-3}\partial_x$$

$$= -\partial_z\left(\tfrac{1}{3} - zwx^{p-1}y^{l-3}\right) \cong -zx^{p-1}y^{l-3}\partial_y + pzwx^{p-2}y^{l-3}\partial_z$$

$$= -\left(s^5+st^4\right)s^{3p-3}t^{l-3}\partial_y + p\left(s^5+st^4\right)3s^4t^3s^{3p-6}t^{l-3}\partial_z$$

$$\cong -\left(s^5+st^4\right)s^{3p-3}t^{l-3}4st^3\partial_z + p\left(s^5+st^4\right)3s^4t^3s^{3p-6}t^{l-3}\partial_z$$

$$= (3p-4)\left(s^{3(p+1)}t^l + s^{3p-1}t^{l+4}\right)\partial_z$$

The first term equals zero in the quotient, as we had already seen. So we get

$$s^{3p+1}t^l\partial_x = (3p-4)\left(s^{3p-1}t^{l+4}\right)\partial_z$$

$$= (5s^4+t^4-5s^4)(3p-4)\left(s^{3p-1}t^l\right)\partial_z$$

$$\cong 3s^2(3p-4)\left(s^{3p-1}t^l\right)\partial_x - 5(3p-4)\left(s^{3(p+1)}t^l\right)\partial_z$$

$$\cong 3(3p-4)\left(s^{3p+1}t^l\right)\partial_x$$

As $3(3p-4)$ is never equal to one, this means that we can only have zero coefficient. Note however that in order to get this result we had to suppose that $l > 2$ and $p > 0$.

Lemma 4.15. *$s^{3p+1}t^l\partial_x$ is trivial for $l > 2$ and $p > 0$.*

We now proceed with deformations of the form $s^{3p+1}t^l\partial_z$:

$$s^{3p+1}t^l\partial_z \cong \tfrac{1}{4}s^{3p}t^{l-3}\partial_y = \tfrac{1}{4}x^p y^{l-3}\partial_y = -\partial_w(-\tfrac{1}{4}wx^py^{l-3})\partial_y$$

$$\cong -\partial_x(-\tfrac{1}{4}wx^py^{l-3})\partial_z = \tfrac{p}{4}wx^{p-1}y^{l-3}\partial_z$$

$$= \tfrac{p}{4}3s^4t^3s^{3p-3}t^{l-3}\partial_z = \tfrac{3p}{4}s^{3p+1}t^l\partial_z$$

Lemma 4.16. *The deformation $s^{3p+1}t^l\partial_z$ is trivial provided that $l > 2$.*

An easy consequence is triviality of deformations of the type $s^{3p+2}t^l\partial_x$ for $l > 2$:

$$s^{3p+2}t^l\partial_x = 3s^2\left(\tfrac{1}{3}s^{3p}t^l\right)\partial_x \cong \tfrac{1}{3}\left(5s^4+t^4\right)s^{3p}t^l\partial_z \cong \tfrac{5}{3}s^{3(p+1)+1}t^l\partial_z$$

Lemma 4.17. *$s^{3p+2}t^l\partial_x$ is trivial for $l > 2$.*

The only case that remains is $s^{3p+2}t^l\partial_z$. This is similar to what we already did.

$$s^{3p+2}t^l\partial_z = \left(5s^4+t^4-t^4\right)\tfrac{1}{5}s^{3(p-1)+1}t^l\partial_z$$

$$\cong \tfrac{3}{5}s^2s^{3(p-1)+1}t^l\partial_x - t^4\tfrac{1}{5}s^{3(p-1)+1}t^l\partial_z = \tfrac{3}{5}s^{3p}t^l\partial_x - \tfrac{1}{5}s^{3(p-1)+1}t^{l+4}\partial_z$$

The first term is obviously trivial, but also the second, as $l + 4 > 2$.

Lemma 4.18. *$s^{3p+2}t^l\partial_z$ is trivial for all $p > 0$.*

Now we have to calculate the exceptional cases excluded in the previous discussion. We start with $s^{3p+2}t^l\partial_x$ and suppose that $p > 1$ but the exponent l might be arbitrary (e.g. $l < 3$)

$$s^{3p+2}t^l\partial_x = (s^5 + st^4 - st^4)s^{3(p-1)}t^l\partial_x = zx^{p-1}y^l\partial_x - s^{3(p-1)+1}t^{l+4}\partial_x$$

The last term vanishes as $p > 1$ and $l + 4 > 2$. So we have

$$s^{3p+2}t^l\partial_x = zx^{p-1}y^l\partial_x = -\partial_z\left(-\tfrac{1}{2}z^2x^{p-1}y^l\right)\partial_x \cong \frac{p-1}{2}z^2x^{p-2}y^l\partial_z$$

$$= \tfrac{p-1}{2}\left(s^{10} + 2s^6t^4 + s^2t^8\right)s^{3p-6}t^l\partial_z$$

$$= \tfrac{p-1}{2}\left(s^{3(p+1)+1}t^l + s^{3(p-2)+2}t^{l+8}\right)\partial_z$$

As we had $p > 2$, the last term vanishes by what we already calculated, so

$$s^{3p+2}t^l\partial_x = \tfrac{p-1}{2}\left(s^{3(p+1)+1}t^l\right)\partial_z = \tfrac{p-1}{10}\left(5s^4 + t^4 - t^4\right)s^{3p}t^l\partial_z$$

$$\cong \tfrac{3(p-1)}{10}s^{3p+2}t^l\partial_x - \tfrac{p-1}{10}t^4s^{3p}t^l\partial_z$$

The last term is zero as usual, so we have $\frac{3(p-1)}{10} = 1$ which is impossible.

Lemma 4.19. *$s^{3p+2}t^l\partial_x$ is trivial for $p > 2$.*

We continue with $s^2t^l\partial_z$ and suppose that $l > 6$:

$$s^2t^l\partial_z = \tfrac{1}{4}st^{l-3}\partial_y = \left(s^5 + st^4 - s^5\right)\tfrac{1}{4}t^{l-7}\partial_y = \tfrac{1}{4}zy^{l-7}\partial_y + s^6t^4\partial_z$$

$$= -\partial_w\left(-\tfrac{1}{4}wzy^{l-7}\right)\partial_y \cong -\tfrac{1}{4}wy^{l-7}\partial_x = -\tfrac{3}{4}s^4y^{l-4}\partial_x$$

$$\cong -\tfrac{1}{4}\left(5s^4 + t^4\right)s^2y^{l-4}\partial_z = -\tfrac{1}{4}s^2y^l\partial_z$$

This means

Lemma 4.20. *The deformation $s^2t^l\partial_z$ is trivial for $l > 6$.*

As we go on, we find that for $p > 3$ by lemma 4.19:

$$s^{3p+1}t^l\partial_z = (5s^4 + t^4 - t^4)\frac{1}{5}s^{3(p-1)}tl\partial_z \cong \frac{3}{5}s^{3(p-1)+2}t^l\partial_x = 0$$

Lemma 4.21. *The deformation $s^{3p+1}t^l\partial_z$ is trivial for $p > 3$.*

Furthermore

$$s^{3p+1}t^l\partial_x = \frac{1}{3}\left(5s^4 + t^4\right)s^{3(p-1)+2}t^l\partial_z = \frac{5}{3}s^{3(p-1)+2}t^{l+4}\partial_z = 0$$

for $p - 1 > 0$ by lemma 4.18.

Lemma 4.22. *The deformation $s^{3p+1}t^l\partial_x$ is trivial for $p > 1$.*

And finally

$$st^l\partial_x = \left(s^5 + st^4 - s^5\right)t^{l-4}\partial_x = zy^{l-4}\partial_x - s^5t^{l-4}\partial_x$$

The second term vanishes for $l > 6$ (lemma 4.17) so

$$st^l\partial_x = zy^{l-4}\partial_x = -\partial_z\left(-\frac{1}{2}z^2y^{l-4}\right)\partial_x \cong 0$$

So

Lemma 4.23. *$st^l\partial_x$ is trivial for $l > 6$.*

We summarize the results in the following table.

$s^{3p}t^l\partial_x = 0\ ;\ \forall p, l$	lemma 4.14	$s^{3p}t^l\partial_z = 0\ ;\ \forall p, l$	lemma 4.14
$s^{3p+1}t^l\partial_x = 0\ ;\ l > 2, p > 2$	lemma 4.15	$s^{3p+1}t^l\partial_z = 0\ ;\ l > 2$	lemma 4.16
$s^{3p+2}t^l\partial_x = 0\ ;\ l > 2$	lemma 4.17	$s^{3p+2}t^l\partial_z = 0\ ;\ p > 0$	lemma 4.18
$s^{3p+2}t^l\partial_x = 0\ ;\ p > 2$	lemma 4.19	$s^2t^l\partial_z = 0\ ;\ l > 6$	lemma 4.20
$s^{3p+1}t^l\partial_x = 0\ ;\ p > 1$	lemma 4.22	$s^{3p+1}t^l\partial_z = 0\ ;\ p > 3$	lemma 4.21
$st^l\partial_x = 0\ ;\ l > 6$	lemma 4.23		

This proves the finite-dimensionality of $T^1_{IsoDef}(\varphi_5)$. But we need to know the exact dimension. Therefore we have to look at linear relations between the remaining monomials. These are

1.	$s\partial_x$	$st\partial_x$	$st^2\partial_x$	$st^3\partial_x$	$st^4\partial_x$	$st^5\partial_x$	$st^6\partial_x$
2.	$s^2\partial_x$	$s^2t\partial_x$	$s^2t^2\partial_x$				
3.	$s^4\partial_x$	$s^4t\partial_x$	$s^4t^2\partial_x$				
4.	$s^5\partial_x$	$s^5t\partial_x$	$s^5t^2\partial_x$				
5.	$s\partial_z$	$st\partial_z$	$st^2\partial_z$				
6.	$s^2\partial_z$	$s^2t\partial_z$	$s^2t^2\partial_z$	$s^2t^3\partial_z$	$s^2t^4\partial_z$	$s^2t^5\partial_z$	$s^2t^6\partial_z$
7.	$s^4\partial_z$	$s^4t\partial_z$	$s^4t^2\partial_z$				
8.	$s^7\partial_z$	$s^7t\partial_z$	$s^7t^2\partial_z$				

Let us consider $s^2t^l\partial_x$ for $l < 3$. We have

$$s^2t^l\partial_x \cong \frac{5}{3}s^4t^l\partial_z$$

so the second and the seventh line are linear dependent. Furthermore

$$s^7t^l\partial_z = \left(5s^4 + t^4 - t^4\right) s^3t^l\partial_z \cong \frac{3}{5}s^5t^l\partial_x$$

It follows that the last line is a multiple of the fourth one. We can thus reduce the table as follows:

1.	$s\partial_x$	$st\partial_x$	$st^2\partial_x$	$st^3\partial_x$	$st^4\partial_x$	$st^5\partial_x$	$st^6\partial_x$
2.	$s^2\partial_x$	$s^2t\partial_x$	$s^2t^2\partial_x$				
3.	$s^4\partial_x$	$s^4t\partial_x$	$s^4t^2\partial_x$				
4.	$s^5\partial_x$	$s^5t\partial_x$	$s^5t^2\partial_x$				
5.	$s\partial_z$	$st\partial_z$	$st^2\partial_z$				
6.	$s^2\partial_z$	$s^2t\partial_z$	$s^2t^2\partial_z$	$s^2t^3\partial_z$	$s^2t^4\partial_z$	$s^2t^5\partial_z$	$s^2t^6\partial_z$

Moreover, we see that for $t > 4$

$$st^l\partial_x = \left(s^5 + st^4 - s^5\right) t^{l-4}\partial_x = zy^{l-4}\partial_x - s^5t^{l-4}\partial_x \cong -s^5t^{l-4}\partial_x$$

so the three entries of the fourth line are multiples of the last three of the first row. In the same manner,

$$s^4t^l\partial_x = \frac{1}{3}\left(5s^4+t^4\right)s^2t^l\partial_z = \frac{1}{3}s^2t^{l+4}\partial_z$$

proving that the third row is a multiple of the last entries of the sixth row. So we can once again reduce the table to

1.	$s\partial_x$	$st\partial_x$	$st^2\partial_x$	$st^3\partial_x$	$st^4\partial_x$	$st^5\partial_x$	$st^6\partial_x$
2.	$s^2\partial_x$	$s^2t\partial_x$	$s^2t^2\partial_x$				
5.	$s\partial_z$	$st\partial_z$	$st^2\partial_z$				
6.	$s^2\partial_z$	$s^2t\partial_z$	$s^2t^2\partial_z$	$s^2t^3\partial_z$	$s^2t^4\partial_z$	$s^2t^5\partial_z$	$s^2t^6\partial_z$

Now it is more or less obvious (and can be indeed verified) that the remaining elements are linearly independent over $\mathbb{K}$ and therefore constitute non-trivial deformations. So we get the final result

$$\dim_{\mathbb{K}}\left(T^1_{IsoDef}(\varphi_5)\right) = 20$$

Summarizing the above results, we obtain

map	$l\left(\frac{\varphi_*\mathcal{O}_{\mathbb{K}^2,0}}{\mathcal{O}_{L,0}}\right)$	$t^1_{Def}(\varphi)$	$t^1_{IsoDef}(\varphi)$	$l\left(\Omega_{\mathbb{K}^2,0/\mathbb{K}^4,0}\right)$
φ_1	3	5	1	2
φ_2	10	19	6	4
φ_3	21	55	15	6
φ_4	36	97	28	8
φ_5	28	77	20	8

This leads to the following conjecture

Conjecture 4.24. *For isotropic mappings from* $(\mathbb{K}^2,0)$ *to* $(\mathbb{K}^4,0)$ *of corank one, the following relation holds true*

$$\delta = \dim_{\mathbb{K}}\left(T^1_{IsoDef}(\varphi)\right) + \dim_{\mathbb{K}}\left(\Omega_{\mathbb{K}^2/\mathbb{K}^4}\right)$$

whenever all of these three dimension are finite.

If the image of φ is the open Whitney umbrella in $\mathbb{K}^4$, this relation is satisfied: $\left(T^1_{IsoDef}(\varphi)\right) = 0$ in this case as we have proved (theorem 4.4) and one shows directly that $\dim_{\mathbb{K}}\left(\Omega_{\mathbb{K}^2/\mathbb{K}^4}\right) = \delta = 1$. Hence one may try to prove the conjecture by a "conservation of number"-argument using corollary 4.12 (see [dJP00] for a description of this principle), that is, one has to show that the modules $\varphi_*\mathcal{O}_{\mathbb{K}^2\times S,0}/\mathcal{O}_{L_S,0}$, $T^1_{IsoDef}(\varphi_S)$ and $\Omega_{\mathbb{K}^2\times S,0/\mathbb{K}^4\times S,0}$, where S is a parameter space and $\varphi_S : \mathbb{K}^2 \times S \to L_S \subset \mathbb{K}^4 \times S$ a deformation of the given map φ, are free (Cohen-Macaulay is sufficient if S is smooth) $\mathcal{O}_S$-modules. This is however not clear at all, therefore, the above statement remains a conjecture.

Appendix A

Deformation Theory

The aim of this large appendix is to give a review of abstract deformation theory as developed by Schlessinger, Artin, Deligne, Millson and others. All facts presented herein are "well-known", but the appropriate references are rather scattered in the literature. Therefore, we tried to put them together here in one place. There are three central notions which we will explain: categories fibred in groupoids, deformation functors and controlling differential graded Lie algebras (dg-Lie algebras for short). The first two are (non-equivalent) ways to formalize a given deformation problem. On the other hand, to any dg-Lie algebra $(L, d, [\ ,\ ])$ one can associate either a category fibred in groupoids over the category of Artin rings (called $\mathbf{Def}_L$) or a functor on the category of Artin rings (called Def_L). For a given deformation problem, one tries to construct an appropriate dg-Lie algebra and to prove the equivalence of the given fibred category with $\mathbf{Def}_L$ (resp. the isomorphy of the given deformation functor with Def_L). This approach encompasses the more classical notions of the tangent space and of an obstruction theory for a functor. However, it might be very hard to find the right dg-Lie algebra and to prove the above equivalence. We describe some basic examples, namely, deformations of complex manifolds, associative algebras and Lie algebras, and, in more detail, a local version of the cotangent complex.

A.1 Formal deformation theory

In this first part we work in a completely abstract setting. We first introduce differential graded Lie algebras and then turn our attention to deformation functors and fibred categories. Finally, we explain the meaning of a "controlling" dg-Lie algebra. We work over an arbitrary field of characteristic zero, denoted by k.

A.1.1 Differential graded Lie algebras

Here the basic definitions and properties of dg-Lie algebras are given. For a more detailed reference, see [Man98].

Definition A.1. *A dg-Lie algebra $L = \oplus_{i\in\mathbb{Z}} L^i$ is a $\mathbb{Z}$-graded vector space together with a differential, that is, a linear map $d : L^i \to L^{i+1}$ satisfying $d^2 = 0$ and a linear bracket*

$$[\ ,\ ] : L^i \times L^j \longrightarrow L^{i+j}$$

such that

- $[a, b] + (-1)^{ij}[b, a] = 0$ *for all* $a \in L^i$ *and* $b \in L^j$.
- $d[a, b] = [da, b] + (-1)^i[a, db]$ *for* $a \in L^i$ *and* $b \in L^j$.
- $[a, [b, c]] = [[a, b], c] + (-1)^{ij}\,[b, [a, c]]$

We remark that the subspace L^0 with the induced bracket is a Lie algebra in the usual sense.

A morphism between dg-Lie-algebras is a morphism of complexes which preserves the bracket. A dg-Lie algebra is called formal if it is isomorphic to its cohomology (viewed as a dg-Lie algebra with trivial differential and induced bracket).

For further use, we also give the related definition of a differential graded algebra.

Definition A.2. *A differential graded algebra (DGA) is a $\mathbb{Z}$-graded vector space $A = \oplus_{i\in\mathbb{Z}} A^i$ together with a differential $d : A^i \to A^{i+1}$ satisfying $d^2 = 0$ and a linear product*

$$\wedge : A^i \times A^j \longrightarrow A^{i+j}$$

such that

- $a \wedge b = (-1)^{ij} b \wedge a$ *for all* $a \in A^i$, $b \in A^j$.
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *for all* $a, b, c \in A$.
- $d(a \wedge b) = da \wedge b + (-1)^i a \wedge db$ *for all* $a \in A^i$, $b \in A$.

Again, a morphism of DGA's is a morphism of complexes commuting with the differentials and respecting the products.

Let us return to dg-Lie algebras.

Definition A.3. *Let* $(L, d, [\ ,\ ])$ *be a dg-Lie algebra. The set* $MC_L \subset L^1$ *(called the set of solutions of the Maurer-Cartan equation) is by definition*

$$MC_L = \left\{ a \in L^1 \mid da + \frac{1}{2}[a, a] = 0 \right\}$$

It is immediate that MC_L *is preserved under a morphism of dg-Lie algebras.*

In order to relate dg-Lie-algebras to deformation problems, we have to find a way to encode the action of an automorphism group on a given set of deformations. Therefore, we will introduce the so-called gauge action. It is known (see, e.g., [Man01b]), that for any (ordinary) Lie algebra $\mathfrak{g}$, there is a group structure on

$$\widehat{\mathfrak{g}} := \varprojlim(\mathfrak{g}/\mathfrak{g}^i)$$

where $\mathfrak{g}^i := [\mathfrak{g}, \mathfrak{g}^{i-1}]$ is the descending central series. If $\mathfrak{g}$ is nilpotent, we get a product on $\mathfrak{g} = \widehat{\mathfrak{g}}$ which is called *Campbell-Baker-Hausdorff-multiplication*. The formula which defines it is somewhat complicated to write down. We note the first terms of the Campbell-Baker-Hausdorff-product $*$:

$$a * b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] - \frac{1}{12}[b, [b, a]] + \ldots$$

For every representation $\rho : \mathfrak{g} \to End(V)$ there is an induced representation of groups $e^\rho : (\mathfrak{g}, *) \to Aut(V)$ satisfying $e^\rho(n) = \exp(\rho(n)) = \sum_{i=0}^{\infty} \frac{1}{i!}\rho(n)^i$.

Lemma A.4. *Let* $(L, d, [\ ,\ ])$ *be a dg-Lie algebra such that* L^0 *is nilpotent. Consider the adjoint action* $\rho : L^0 \to End(L^1)$ *where* $\rho(n)(v) = [n, v]$. *Then the (conical) set*

$$\left\{v \in L^1 \,|\, [v, v] = 0\right\}$$

is invariant under the exponential action e^ρ.

Proof. See [Man98]. □

We want to show that not only the set $\left\{v \in L^1 \,|\, [v, v] = 0\right\}$, but even MC_L is invariant under the action e^ρ. This can be done in an elegant way as follows: For a given dg-Lie algebra $(L, d, [\ ,\ ])$, consider the following $\mathbb{Z}$-graded k-vector space:

$$L_d := \oplus_{i \in \mathbb{Z}} L_d^i$$

where $L_d^i := L^i$ for $i \neq 1$ and $L_d^1 := L^1 \oplus kd$. Define a dg-Lie-algebra structure on L_d by

$$\begin{aligned} d_d(a + cd) &:= d(a) \\ [a + c_1 d, b + c_2 d]_d &:= [a, b] + c_1 d(b) + (-1)^i c_2 d(a) \end{aligned}$$

for $a \in L^i, b \in L^j$ and $c, c_1, c_2 \in k$. Then we see that the mapping

$$\begin{aligned} \Phi : L &\longrightarrow L_d \\ v &\longmapsto d + v \end{aligned}$$

is a morphism of dg-Lie-algebras and that $a \in L^1$ is a solution of the Maurer-Cartan equation iff $[\Phi(v), \Phi(v)]_d = 0$. We can now apply lemma A.4 to the dg-Lie-algebra $(L_d, d_d, [\ ,\ ]_d)$. It is obvious that the action e^ρ preserves the affine hyperplane $\left\{v + d \,|\, v \in L^1\right\}$. But the set MC_L is in bijection with the intersection of this affine hyperplane with the cone $\left\{v \in L^1 \,|\, [v, v] = 0\right\}$, so we get that the action e^ρ preserves MC_L.

A.1.2 Categories fibred in groupoids and deformation functors

Fibred categories are a very general setup to discuss any type of deformation problems. We do not give the lengthy definition here (see

[BF96]), but only note that a fibred category is a functor $p : \mathbf{F} \to \mathbf{C}$ satisfying properties concerning the pullback of an object $f \in F$ by a morphism $(A \to p(f)) \in Mor(C)$. It follows that the *fibre* $F(A)$ is a category. Given a fibred category, one can associate a canonical functor from $\mathbf{C}$ to **Sets** which sends $A \in \mathbf{C}$ to the set of isomorphism classes of objects in $F(A)$. It is also possible to construct a fibred category from such a functor, but this category will differ from the original one, namely, by passing from a fibred category to the associated functor one loses the information contained in the automorphisms of the fibre categories. Most of the fibred categories found in deformation theory have a special property: The fibre categories are groupoids, i.e., there are only isomorphisms over the identity morphism of an object $A \in \mathbf{C}$. In that case one says that $\mathbf{F}$ is a *category fibred in groupoids.* In principle it is more appropriate to work with categories fibred in groupoids than with the associated functors. However, the latter approach is simpler and sufficient for our purpose. We will therefore restrict ourselves to a description of the theory of functors associated to deformation problems. We will make an additional assumption in the sequel: The category $\mathbf{C}$ will be assumed to be the category of Artin rings (or its opposite category). In that case one can study deformation problems only in the formal sense, that is, statements like existence of versal deformations, triviality of given deformations etc. will always be statements on algebras or modules over formal power series rings. How to obtain convergent solutions is a completely different issue. We will not treat it here, one might consult [dJP00] for a description of some techniques involving approximation theorems.

The classical reference for the theory of functors on Artin rings is [Sch68], where conditions for a functor to have a hull (i.e., a formally versal deformation) are given. Schlessinger introduced the vector space T^1_F called tangent space of the functor F and the most important of the above conditions is that its dimension is finite. More recently, Fantechi and Manetti described in [FM98] a similar formalism for obstructions, that is, they associate to a deformation functor a vector space called T^2_F which contains obstructions to the extension of a given deformation to a larger space. In the case that the deformation problem is governed by a dg-Lie algebra $(L, d, [\ ,\ ])$ (we will define what this means), these spaces are simply the first and second cohomology of L. The meaning of the higher cohomology groups is less obvious, but can apparently be understood using the concept of extended deformation functors (see [Man99] and [BK98]).

Consider the category **Art** of local Artin rings with residue field k and the category $\widehat{\mathbf{Art}}$ of complete local (noetherian) rings with residue field k. We call short exact sequences

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

in **Art** small extensions of A by M iff $\mathbf{m}_B M = 0$. Small extensions with one-dimensional kernels, that is, sequences of the form

$$0 \longrightarrow k \longrightarrow B \longrightarrow A \longrightarrow 0$$

are called principal small extensions.

Let Set be the category of pointed sets with distinguished element $*$. Then we consider functors from **Art** to **Set** such that $F(k) = *$. Such functors together with natural transformations form a category which is called Fun in [Man98]. There are special morphisms in Fun.

Definition A.5. *Let $\nu : F \to G$ be a natural transformation of functors (i.e., a morphism in in Fun). Then we will call ν:*

- ***smooth*** *iff for any surjection $A' \to A$ the canonical map*

$$F(A') \longrightarrow G(A') \times_{G(A)} F(A)$$

is surjective. A functor $F \in Fun$ is called smooth if the morphism $F \to \{*\}$ to the constant functor (the final object in the category Fun) is smooth.

- ***unramified****, if the induced morphism on tangent spaces*

$$T^1_F := F(k[\epsilon]) \longrightarrow T^1_G := G(k[\epsilon])$$

is injective.

- ***étale*** *iff it is smooth and unramified (Note that then ν is automatically bijective on tangent spaces, these morphisms are also called minimally smooth).*

Now we can characterize functors which admit universal or at least versal deformation spaces.

Definition A.6. *A functor $F \in Fun$ is called pro-representable iff there exists $R \in \widehat{\mathbf{Art}}$ such that F is isomorphic to the functor $h_R : Art \to Set$ defined by $h_R(A) := Hom(R, A)$ via the natural transformation*

$$\begin{array}{rcl} PB_R : h_R & \longrightarrow & F \\ (\Phi : R \to S) & \longmapsto & F(\Phi) \end{array}$$

(PB stands for pull-back). R is called a hull iff the morphism PB_R is only étale.

Note that the tangent space of a functor having a hull R is canonically identified with the Zariski tangent space $\left(\mathbf{m}_R/\mathbf{m}_R^2\right)^*$ of $Spec(R)$.

Schlessinger introduced conditions for a functor to be pro-representable or to have a hull. We list here these properties together with some modifications which can be found [FM98].

Definition A.7. *Let $F \in Fun$ and $A' \to A$ and $A'' \to A$ be morphism in* ***Art****, the latter being a small extension. Consider the canonical map*

$$\eta_{A',A'',A} : F(A' \times_A A'') \longrightarrow F(A) \times_{F(A)} F(A'')$$

Then we have the following conditions for the functor F:

(H1) the map $\eta_{A',A'',A}$ is surjective for all small extensions $A'' \to A$.

(H2) $\eta_{A',A'',A}$ is bijective for $A = k$, $A'' = k[\epsilon]$. A functor satisfying (H1) and (H2) is called deformation functor.

(H2') $\eta_{A',A'',A}$ is bijective for $A = k$ and arbitrary A''. Such a functor is called deformation functor with obstruction theory (see section A.1.3 on page 143).

(H3) the ***tangent space*** *T_F^1 of F is finite-dimensional over k. (Note that $H2$ guarantees that T_F^1 is a vector space.)*

(H4) The map $\eta_{A',A'',A}$ is bijective for every small extension $A'' \to A$. A functor satisfying this condition is also called ***homogeneous****.*

We now reproduce the fundamental theorem from [Sch68] which justifies the above conditions.

Theorem A.8. *Let $F \in Fun$ be a deformation functor with finite-dimensional tangent space ((H1), (H2) and (H3) are satisfied). Then there is a hull $R \in \widehat{\mathbf{Art}}$. If, in addition, (H4) holds, then R pro-represents F.*

Proof. We follow the proof in [Art76]. A hull in the above sense is a complete ring $R \in \widehat{\mathbf{Art}}$, together with elements $X_n \in F(R_n)$ where $R_n := R/\mathbf{m}_R^{n+1}$ such that $\mathcal{O}_{X_n} \otimes_{R_n} k = \mathcal{O}_{X_0}$ and $\mathcal{O}_{X_n} \otimes_{R_n} R_{n-1} = \mathcal{O}_{X_{n-1}}$ for all n (X_0 is the unique object in $F(k)$) and such that for all X_n the versality condition holds in the subcategory $\mathbf{Art}_n$ of rings $P \in \mathbf{Art}$ with $\mathbf{m}_P^{n+1} = 0$. We proceed by induction on n. For $n = 1$, choose a basis of $\epsilon_1, \ldots, \epsilon_\tau$ of T_F^1 and consider $S = k[\epsilon_1, \ldots, \epsilon_\tau]$ and $R_1 = S/\mathbf{m}_S^2$. Set $X_1 := k \oplus T_F^1$. Then X_1 is versal over R_1. Now suppose that a versal X_{n-1} over R_{n-1} is constructed. Suppose R_{n-1} to be a quotient of S by an ideal J_{n-1}. Consider the following set

$$\mathcal{S} := \Big\{ I \subset S \quad | \quad I \subset J_{n-1};\ \mathbf{m}_S J_{n-1} \subset I;\ \exists X_I \in F(I);$$

$$\mathcal{O}_{X_I} \otimes_{S/I} R_{n-1} = \mathcal{O}_{X_{n-1}} \Big\}$$

This set is closed under intersections: As $S/(I_1 \cap I_2) = S/I_1 \times_{S/J_{n-1}} S/I_2$, we see by the axiom (H1) that any two deformations over S/I_1 and S/I_2 are liftable to a deformation over $S/(I_1 \cap I_2)$. Therefore, there is a minimal element, which we denote by J_n. Define $R_n := S/J_n$ and $X_n := X_I$ (one can take any X_I over R_n here that lifts X_{n-1}). It remains to check that X_n/R_n is versal which amounts to show that the transformation

$$\begin{array}{rcl} PB_{R_n} : h_{R_n} & \longrightarrow & F \\ (\Phi : R_n \to A) & \longmapsto & \Phi^* F(A) \end{array}$$

(of functors on $\mathbf{Art}_n$) is smooth. Suppose that we are given a surjection $A' \to A$ in $\mathbf{Art}_n$, a morphism $X_{A'} \to X_A$ over $A' \to A$ and $\Phi : R_n \to A$. Then we have to find a lift $R_n \to A'$ such that $\mathcal{O}_{X_{A'}} = \mathcal{O}_{X_n} \otimes_{R_n} A'$. It is in fact sufficient to do it only for small extensions $A' \to A$, and even only for principal small extensions. So suppose that $ker(A \to A')$ is of dimension one. Denote by R' the fibre sum ring $R' := R_n \times_A A'$. Then by (H1), there is a deformation $X_{R'}$ over R' restricting to X_n over R_n and to $X_{A'}$ over A'. By smoothness of S, we can lift the morphism $S \twoheadrightarrow R_n$ to R'. But the image of $S \to R'$ and of $S \twoheadrightarrow R_n$ coincides, due to the minimality of J_n. This yields a splitting $R_n \to R'$ of the morphism $R' \to R_n$ and we can write $R' \cong R_n \times_k k[\epsilon]/\epsilon^2$, where the isomorphism depends on a chosen homomorphism from $Hom(R_n, k[\epsilon]/\epsilon^2)$. For each such isomorphism $R' \cong R_n \times_k k[\epsilon]/\epsilon^2$, we get an induced deformation $\widetilde{X}_{R'} := \mathcal{O}_{X_{R_n}} \otimes_{R_n} R'$. On the other hand, $Hom(R_n, k[\epsilon]/\epsilon^2) = Hom(R_1, k[\epsilon]/\epsilon^2) = T^1_F$, so $\widetilde{X}_{R'}$ depends on the choice of an element from T^1_F. Axiom (H2) tells us that $F(R') \cong F(R_n) \times_{F(k)} T^1_F$. Obviously, $\widetilde{X}_{R'}$ and $X_{R'}$ both projects on X_{R_n} over R_n. Then by taking their difference in T^1_F as the homomorphism defining the identification $R' \cong R_n \times_k k[\epsilon]/\epsilon^2$, we get alter $\widetilde{X}_{R'}$ to become isomorphic to $X_{R'}$. Then we have $X_{R_n} \otimes_{R_n} R' = \widetilde{X}_{R'} = X_{R'}$ and $X_{R'} \otimes'_R A' = X_{A'}$, so the composition map $R_n \to R' \to A'$ (the first one is the splitting from above, the second the projection) is the required morphism satisfying $\mathcal{O}_{X_{R_n}} \otimes_{R_n} A' = \mathcal{O}_{X_{A'}}$. The proof of the second statement will not be given here. It can be found in [Sch68]. $\square$

A.1.3 Obstruction theory

From the previous section we know that functors $F \in Fun$ satisfying Schlessinger's conditions admit a hull R. But this does not give any information on the structure of the space $Spec(R)$. In particular, we do not know whether it is smooth or not. Obstruction theory is concerned with this question. More specifically, one asks whether for a given small extension

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

the induced map $F(B) \to F(A)$ is surjective. Note that this is nothing else but the fact that the functor F is smooth in the sense of definition A.5 on page 140.

The most general treatment of obstruction theory is found in [FM98]. In that paper, obstructions are not defined for a single element $F \in Fun$ but rather for a morphism $F \to G$ of deformation functors and consequently called relative obstruction theories. However, in our applications this generality will not be needed. Therefore, we will restrict ourselves to the theory described in [Man98].

Definition A.9. *Let $F \in Fun$, then an* ***obstruction theory*** *of F, denoted by (V, v_F) consists of the following data:*

- *a k-vector space V*
- *a map $v_F(e) : F(A) \to V \otimes_k M$ associated to any small extension*

$$e : 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

such that the following properties are satisfied:

1. *Let $\eta \in F(A)$ given, such that η lies in the image of the map $F(B) \to F(A)$. Then $v_F(e)(\eta) = 0$*
2. *Let $\alpha : e_1 \to e_2$ be a morphism of small extensions, i.e.:*

$$\begin{array}{ccccccccc} e_1 : & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\ e_2 : & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

   and $\eta \in F(A_1)$ then

$$v_F(e_2)\left(F(\alpha_A)(\eta)\right) = \left(Id_V \otimes \alpha_M\right)\left(v_F(e_1)(\eta)\right)$$

An obstruction theory for which the converse of 1. holds is called ***complete****. Morphisms of obstruction theories are defined in the obvious way: as a map of vector spaces $\varphi : V \to V'$ such that $v'_F(e) = \varphi \circ v_F(e)$. Then an obstruction theory (O, v_F) is called universal iff it is "the smallest one", i.e., if there is an unique morphism $(O_F, v_F) \to (V, v_F)$ for any other given obstruction theory (V, v_F) of the functor F.*

A major result in [FM98] is that a functor $F \in Fun$ satisfying the conditions (H1) and (H2') (which were called deformation functors with obstruction theory in the above definition) does indeed have a universal obstruction theory which is complete and consists only of obstructions associated to principal extensions. However, the proof is rather abstract and does not give much advice how to construct a universal obstruction theory for a given deformation functor.

As a first application, we give conditions for functors and morphisms to be smooth.

Theorem A.10. *Let* $\nu : F \to G$ *be a morphism of functors and* (V, v_F), (W, v_G) *obstruction theories for* F *and* G*, respectively, then we call a linear map* $v_\nu : V \to W$ *compatible iff for each small extension* $0 \to M \to B \to A \to 0$ *and each* $\eta \in F(A)$ *we have* $(v_G \circ \nu)(\eta) = (v_\nu \otimes Id_M) \circ v_F(\eta)$. *Then the following holds:* ν *is smooth if* (V, v_F) *is complete,* v_ν *is injective and* $T^1_F \to T^1_G$ *is surjective.*

Proof. First we prove the following preliminary fact: For any functor $F \in Fun$ and any small extension as in the theorem, there is a natural *transitive* action of $T^1_F \otimes M$ on the fibres of $F(B) \to F(A)$. For this one first needs to identify $F(k \oplus M)$ ($k \oplus M$ being the trivial extension of M by k) with $T^1_F \otimes M$ which is easily done by induction on the length of B. Then consider $C := B \times_A B$. We have $C \cong B \times_k (k \oplus M)$ so, by (H2')

$$F(C) = F(B) \times (T^1_F \otimes M)$$

From the natural morphism $\alpha : F(C) \to F(B) \times_{F(A)} F(B)$ we obtain a map

$$F(B) \times (T^1_F \otimes M) \longrightarrow F(B) \times_{F(A)} F(B)$$

which by construction is the identity on the first factor. Composing with the second projection, we get finally a map $F(B) \times (T^1_F \otimes M) \longrightarrow F(B)$ which induces the group action we are looking for. Transitivity follows immediately from the surjectivity of α which comes from condition (H1).

Let an element $(a, b') \in F(A) \times_{G(A)} G(B)$ be given. Our task is to find $b \in F(B)$ which projects to $a \in F(A)$ and $b' \in G(B)$. Denote by $a' \in G(A)$ the common image of a and b' in $G(A)$. As b' is a lift of a' to $G(B)$, we have that $v_G(a') = 0 \in W \otimes M$. By compatibility and injectivity of v_ν we get $v_F(a) = 0$ in $V \otimes M$. But (V, v_F) is complete, so we can find $\widetilde{b} \in F(B)$ lifting $a \in F(A)$. It is not true that the image $\widetilde{b'} = \nu(\widetilde{b})$ is equal to b'. But as $(\widetilde{b'}, b')$ is in $G(B) \times_{G(A)} G(B)$, we find

$t' \in T^1_G \otimes M$ which sends $\widetilde{b}'$ to b'. By surjectivity of $T^1_F \to T^1_G$ there is $t \in T^1_F \otimes M$ which can be used to find an element b lying in the same fibre of $F(B) \to F(A)$ as $\widetilde{b}$ and having the desired properties. □

For any morphism $\nu : F \to G$ of functors and for any obstruction theory (W, v_G) of G, the composition $(W, v_G \circ \nu)$ is an obstruction theory for F. By taking $W = O_G$ and using the universality of O_F we obtain a linear map $O_F \to O_G$. Applying the preceding theorem yields:

Corollary A.11. *Let $\nu : F \to G$ be a morphism and consider the universal obstruction theories O_F and O_G.*

- *ν is smooth iff $T^1_F \to T^1_G$ is surjective and $O_F \to O_G$ is injective.*
- *F is smooth iff $O_F = 0$*

Proof. It remains to prove that for a smooth morphism ν the map $o_\nu : O_F \to O_G$ is injective. So suppose that there is an $x \in O_F$ such that $o_\nu(x) = 0$. By universality, there is a small extension $B \twoheadrightarrow A$ and $\eta \in F(A)$ such that $v_F(\eta) = x$. As O_G is complete, we can lift $\nu(\eta) \in G(A)$ to $G(B)$. But then by smoothness of ν there is a lift of η to $F(B)$ which in turn implies that $v_F(\eta) = x$ vanishes. □

The universal obstruction theory of a pro-representable functor can be explicitly described. First remark that for each small extension $e : 0 \to M \to B \to A \to 0$ and morphisms $\phi : A' \to A$ resp. $\psi : M \to M'$ we have a pullback $\phi^* e$ and a pushforward $\psi_* e$ defined as follows: $\phi^* e$ is the extension

$$0 \longrightarrow M \longrightarrow A' \times_A B \longrightarrow A' \longrightarrow 0$$

whereas $\psi_* e$ is

$$0 \longrightarrow M' \longrightarrow B' \longrightarrow A \longrightarrow 0$$

with $B' := (B \oplus M') \,/\, (\{m, \psi(m) \,|\, m \in M\})$.

Theorem A.12. *Let $R = P/I$ where $P = k[[x_1, \ldots, x_n]]$ and $I \subset \mathbf{m}_P^2$. Then we have the small extension*

$$u_R : 0 \longrightarrow I/\mathbf{m}_P I \longrightarrow P/\mathbf{m}_P I \longrightarrow R \longrightarrow 0$$

and the universal obstruction space of the functor pro-represented by R is $O_{h_R} := (I/\mathbf{m}_P I)^*$.

Proof. Define the obstruction map v_{h_R} as follows: Let

$$e : 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

be any small extension and $\eta \in h_R(A)$. This induces a morphism $\eta : P \to A$. Choose any lift to a morphism $\widetilde{\eta} : P \to B$. Obviously, $\widetilde{\eta}(I) \subset M$ and $\widetilde{\eta}$ maps $\mathbf{m}_P$ to $\mathbf{m}_B$. Therefore, $\eta(\mathbf{m}_P I) = 0 \in B$ and we obtain a map $P/\mathbf{m}_P \to B$ which in turn induces the map

$$\lambda_\eta : I/\mathbf{m}_P I \longrightarrow M$$

Then define $v_{h_R}(\eta) := \lambda_\eta \in (I/\mathbf{m}_P I)^* \otimes M$. We see that λ_η is zero iff $\eta(I) = 0 \in B$. This means that there is a lift of η to B showing that we have a well-defined obstruction theory. That it is indeed universal is proved in [FM98].

We note that using the above definitions of pullback and pushforward, we could have defined λ_η as the element of $(I/\mathbf{m}_P I)^* \otimes M = Hom(I/\mathbf{m}_P I, M)$ such that $\eta^* e = \lambda_{\eta_*} u_R$. □

We now introduce a concept which will be important in the next section, where functors canonically associated to any dg-Lie algebra will be considered. We will call a functor G a group functor if the composition with the forgetful functor from Groups to Sets is an object of Fun. We will suppose that G is smooth (meaning that it is smooth viewed as an object of Fun). Then for a given deformation functor $F \in Fun$ we say that G acts on F iff there is for each $A \in \mathbf{Art}$ a morphism

$$G(A) \times F(A) \longrightarrow F(A)$$

which is a group action in the usual sense. Moreover, we require these actions to be compatible with morphisms in **Art**.

Lemma A.13. *Consider the action*

$$* : T^1_G \times T^1_F \longrightarrow T^1_F$$

and the induced map $\nu : T^1_G \to T^1_F$, given by $\nu(g) = g * 0$. Then we have:

1. $(g + h) * (a + b) = (g * a) + (h * b)$ *and* $t(g * a) = (tg) * (ta)$ *for all* $g, h \in T^1_G$, $a, b \in T^1_F$ *and* $t \in k$.

2. ν is linear.

3. $g * v = \nu(g) + v$.

Proof. The first point is clear from the definition since the structure of a vector space of T^1 is defined using morphisms in **Art**. Then by setting $a = b = 0$ in the formulas in 1. we get that ν is linear and by setting $h = 0$ and $a = 0$ we obtain the formula in 3. □

In this situation, one can consider the *quotient functor* $D := F/G$ which associates to $A \in$ **Art** the set of orbits of $F(A)$ under the action of $G(A)$. Then we have an obvious morphism $F \to D$ in Fun.

Theorem A.14. *D is a deformation functor and the projection $F \to D$ is smooth. We have $T^1_D = coker\left(\nu : T^1_G \to T^1_F\right)$. The group action of G on any obstruction theory (V, v_F) is trivial. In particular, there is an isomorphism $O_F \to O_D$.*

Proof. The first two parts follow immediately from the definitions. Lemma A.13 on the preceding page describes the action of G on F on the infinitesimal level and yields $T^1_D = coker(\nu)$. The statement on obstructions then follows from theorem A.10 on page 145. □

A.1.4 The functors MC_L, G_L and Def_L

We are now in the position to describe the precise relation between dg-Lie algebras and deformation functors.

Definition A.15. *Let $(L, d, [\ ,\ ])$ be a dg-Lie algebra. Then we define*

- *The gauge functor G_L :* **Art** $\to$ **Groups***, defined as:*

$$G_L(A) := \exp(L^0 \otimes \mathbf{m}_A)$$

- *The Maurer-Cartan functor MC_L :* **Art** $\to$ **Sets***:*

$$MC_L(A) := MC_L(L \otimes \mathbf{m}_A) = \left\{ x \in L^1 \otimes \mathbf{m}_A \,|\, dx + \frac{1}{2}[x, x] = 0 \right\}$$

- *The deformation functor Def_L which is by definition the quotient of MC_L by G_L. Remember that the action of G_L on MC_L was defined in section A.1.1 using the fact that $L^0 \otimes \mathbf{m}_A$ is nilpotent.*

Lemma A.16. *Tangent and obstruction spaces of the above functors are as follows.*

1. *G_L is smooth with tangent space $T^1_{G_L} = L^0 \otimes k\epsilon$.*
2. *$T^1_{MC_L} = Z^1(L) \otimes k\epsilon$ where we use the notations $Z^i(L) = ker(d : L^i \to L^{i+1})$ and $B^i(L) = \mathrm{Im}(d : L^{i-1} \to L^i)$.*
3. *A complete obstruction theory for MC_L is given by $\big(H^2(L), v_{MC_L}\big)$, where v_{MC_L} will be defined in the proof.*
4. *The **primary obstruction map** of the functor MC_L, i.e., the obstruction map associated to the small extension*
$$0 \longrightarrow k\epsilon \longrightarrow k[\epsilon]/(\epsilon^3) \longrightarrow k[\epsilon]/(\epsilon^2) \longrightarrow 0$$
is given by $Z^1 \to H^2$, $x \mapsto \frac{1}{2}[x, x]$.
5. *$T^1_{Def_L} = H^1(L)$. As for MC_L, H^2 is a complete obstruction space with primary obstruction map $H^1 \to H^2$, $x \mapsto \frac{1}{2}[x, x]$.*

Proof.

1. The smoothness of G_L is obvious, as we have a surjective group homomorphism $\exp(L^0 \otimes \mathbf{m}_B) \twoheadrightarrow \exp(L^0 \otimes \mathbf{m}_A)$ for any small extension $B \twoheadrightarrow A$. The tangent space of G_L (as a vector space) is by definition $L^0 \otimes \mathbf{m}_{k[\epsilon]/(\epsilon^2)} = L^0 \otimes k\epsilon$.
2. Recall that the Lie bracket on a tensor product of a (graded) Lie algebra with an associative algebra is defined as the Lie bracket on the terms coming from the Lie algebra times the ordinary product on the other terms. This implies that for an element x of $L \otimes k\epsilon$, the bracket $[x, x]$ is automatically zero. Therefore, $MC_L(k[\epsilon]/(\epsilon^2)) = Z^1(L) \otimes k\epsilon$.
3. We first have to define the obstruction map v_{MC_L}. Consider a small extension in **Art**:
$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

Let $x \in MC_L(A)$ be given. Then choose a lift $\widetilde{x} \in L^1 \otimes \mathbf{m}_B$. Define $h := d\widetilde{x} + \frac{1}{2}[\widetilde{x}, \widetilde{x}]$. As $\widetilde{x}$ projects to $x \in A$ and $dx + \frac{1}{2}[x, x] = 0$ in A we see that $h \in L^2 \otimes M$. Then

$$dh = dd\widetilde{x} + [d\widetilde{x}, \widetilde{x}] = \left[h - \frac{1}{2}[\widetilde{x}, \widetilde{x}], \widetilde{x}\right] = [h, \widetilde{x}] - \frac{1}{2}\,[[\widetilde{x}, \widetilde{x}], \widetilde{x}]$$

By the graded Jacobi identity, $[[\widetilde{x}, \widetilde{x}], \widetilde{x}] = 0$. But the first term also vanishes, because $\left[L^2 \otimes M, L^1 \otimes \mathbf{m}_B\right] = 0$ (remember that $\mathbf{m}_B M = 0$). So $h \in Z^2(L) \otimes M$ and we define $v_{MC_L}(x)$ to be the class of h in $H^2(L) \otimes M$. It is clear from the construction that the obstruction class $v_{MC_L}(x)$ is independent of the choice of the lifting $\widetilde{x}$. Indeed, any other lift is given by $\widetilde{x} + z$ with $z \in L^1 \otimes M$. Then $h = d\widetilde{x} + \frac{1}{2}[\widetilde{x}, \widetilde{x}] + dz$ as $[z, z] = [z, \widetilde{x}] = 0$ (because $M \cdot M \subset \mathbf{m}_A M = 0 \in B$). So the class of h in $H^2(L) \otimes M$ is well-defined. Now we have to show that (H^2, v_{MC_L}) is a complete obstruction theory. One part is easy: Given $x \in MC_L(A)$ which lifts to $y \in MC_L(B)$ then $v_{MC_L}(x) = 0$, just take $\widetilde{x} = y$. Conversely, suppose that $v_{MC_L}(x) = 0$. Then there is $z \in L^1 \otimes M$ with $d\widetilde{x} + \frac{1}{2}[\widetilde{x}, \widetilde{x}] = dz$. Set $y := \widetilde{x} - z$. By the same argument as above we get that y is in $MC_L(B)$ thus defining a lift of x.

4. Let an x be an element in $T^1_{MC_L} = Z^1(L) \otimes \mathbf{m}_{k[\epsilon]/(\epsilon^2)}$, then the lift $\widetilde{x} \in L^1 \otimes \mathbf{m}_{k[\epsilon]/(\epsilon^3)}$ can be chosen to lie in $Z^1(L) \otimes \mathbf{m}_{k[\epsilon]/(\epsilon^3)}$. Therefore, the obstruction is simply $\frac{1}{2}[\widetilde{x}, \widetilde{x}] = \frac{1}{2}[x, x] \in H^2(L) \otimes k\epsilon^2$.

5. The action of $T^1_{G_L}$ on $T^1_{MC_L}$ is easy to describe: Let $x \in Z^1(L) \otimes k\epsilon$ and $a \in L^1 \otimes k\epsilon$ be given, then, by definition, the action of a is given as an action $e^{ad(a)}$ on $L^1_d \otimes k\epsilon$ preserving the hyperplane $\{d + x \,|\, x \in L^1 \otimes k\epsilon\}$, namely

$$\begin{aligned} e^{ad(a)}(d + x) &= \left((d + x) + [a, d + x]_d + \tfrac{1}{2}\,[a, [a, d + x]_d]_d + \ldots\right) \\ &= (d + x + [a, d + x]_d) = (d + x + da_d) \end{aligned}$$

So the action $T^1_{G_L} \to End(T^1_{MC_L})$ is simply

$$a \longmapsto (x \mapsto x + da)$$

So we see that $T^1_{Def_L} = H^1(L)$. It follows from theorem A.14 on page 148 that $(H^2(L), v_{Def_L})$ with $v_{Def_L}(x) := v_{MC_L}(x')$ where x' is a lift of $x \in Def_L(A)$ to $MC_L(A)$ is a complete obstruction theory. From the last point we see that the primary obstruction map is

$$\begin{array}{rcl} H^1(L) & \longrightarrow & H^2(L) \\ x & \longmapsto & \frac{1}{2}[x,x] \end{array}$$

□

Suppose now that we are give a morphism $\phi : L \to K$ of dg-Lie algebras (we would like to stress the fact that this is a morphism of complexes which is compatible with the brackets). Obviously, this induces morphisms of functors $\phi_G : G_L \to G_K$ and $\phi_{MC} : MC_L \to MC_K$ which are compatible in the sense that the diagram

$$\begin{array}{ccc} G_L \times MC_L & \longrightarrow & MC_L \\ \Big\downarrow{\scriptstyle \phi_G \times \phi_{MC}} & & \Big\downarrow{\scriptstyle \phi_{MC}} \\ G_K \times MC_K & \longrightarrow & MC_K \end{array}$$

commutes. So we have a morphism of deformation functors $Def_L \to Def_K$.

Theorem A.17. *If $\phi : H^1(L) \to H^1(K)$ is bijective and $\phi : H^2(L) \to H^2(K)$ is injective, then $Def_L \to Def_K$ is étale. If moreover $\phi : H^0(L) \to H^0(K)$ is surjective, then $Def_L \to Def_K$ is an isomorphism.*

Proof. The first statement follows directly from the smoothness criterion above (theorem A.10 on page 145). The second one is a bit more involved and requires a careful analysis of the action of G_L on MC_L. A proof can be found in [Man98]. □

Now we obtain the following fundamental result as an easy consequence.

Corollary A.18. *Suppose that $\phi : L \to K$ is a quasi-isomorphism. Then Def_L and Def_K are isomorphic.*

A.1.5 The T^1-lifting property

The ideas that we will present in this section are essentially due to Z. Ran, see e.g. [Ran92]. However, we will rather follow the paper [Gro97] (Note that a more general version of what follows is proven in [FM99]). The T^1-lifting property is a criterion which ensures the smoothness of a functor by studying relative versions of its tangent space. Originally, this was used to prove that the moduli space of deformations of a Calabi-Yau manifold (the functor of deformations of its complex structure) is smooth. We reproduce this argument in section A.2.1 on page 156 to illustrate the T^1-lifting criterion.

We first consider the general situation of definition A.7 on page 141 and introduce an additional condition for a functor in Fun.

Definition A.19. *Let $F \in Fun$ be a deformation functor. Then we say that condition (H5) holds iff for each pair of surjections $A' \to A$ and $A'' \to A$ we have a map*

$$\tau_{A',A'',A} : F(A') \times_{F(A)} F(A'') \longrightarrow F(A' \times_A A'')$$

such that $\eta_{A',A'',A} \circ \tau_{A',A'',A}$ is the identity on $F(A') \times_{F(A)} F(A'')$ and such that the following holds: Consider a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

This induces morphisms $\varphi_1 : F(B) \to F(A' \times_A A'')$ and $\varphi_2 : F(B) \to F(A') \times_{F(A)} F(A'')$. Then we require that $\varphi_1 = \tau_{A',A'',A} \circ \varphi_2$.

This condition is in some sense a relative version of the above condition (H2). More precisely, let us use the following abbreviations

$$\begin{aligned} A_n &:= k[\epsilon]/(\epsilon^{n+1}) \\ B_n &:= k[x,y]/(x^{n+1}, y^2) \\ C_n &:= k[x,y]/(x^{n+1}, y^2, x^n y) \end{aligned}$$

let $\alpha_n : A_{n+1} \to A_n$, $\beta_n : B_n \to A_n$, $\xi_n : B_n \to B_{n-1}$ and $\gamma_n : B_n \to C_n$ the natural morphisms, set

$$\begin{array}{rcl} \pi_n : A_{n+1} & \longrightarrow & B_n \\ \epsilon & \longmapsto & x+y \end{array}$$

$$\begin{array}{rcl} \pi'_n : A_n & \longrightarrow & C_n \\ \epsilon & \longmapsto & x+y \end{array}$$

and define

$$T^1_{X_n/A_n} := \{Y_n \in F(B_n) \,|\, F(\beta_n)(Y_n) = X_n\}$$

Then (H5) can be used to show that $T^1_{X_n/A_n}$ is an A_n-module: A pair α, β of elements from $T^1_{X_n/A_n}$ lies naturally in $F(B_n) \times_{F(A_n)} F(B_n)$. Therefore it is mapped to $F(B_n \times_{A_n} B_n)$ by τ_{B_n,B_n,A_n} and then to $F(B_n)$ by the underlying map $B_n \times_{A_n} B_n \to B_n$. With all these notations, we can state the main theorem on the T^1-lifting criterion.

Theorem A.20. *Let $F \in Fun$ be a deformation functor with a complete obstruction theory (V, v_F) and which satisfies condition (H5). Pick an element $X_n \in F(A_n)$. Let $X_{n-1} := F(\alpha_{n-1})(X_n)$ be the restriction. Put*

$$Y_{n-1} := F(\pi_{n-1})(X_n) \in T^1_{X_{n-1}/A_{n-1}} \subset F(B_{n-1})$$

Then there is $X_{n+1} \in F(A_{n+1})$ lifting X_n iff Y_{n-1} lies in the image of the canonical restriction mapping $T^1_{X_n/A_n} \to T^1_{X_{n-1}/A_{n-1}}$.

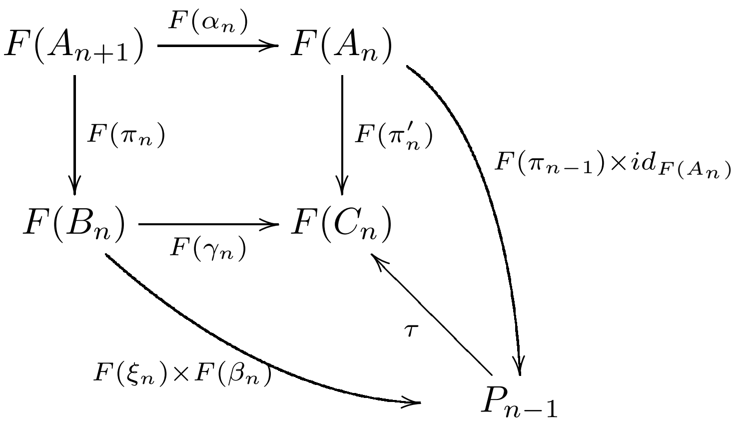
Proof. As $\beta_n \circ \pi_n = \alpha_n$, we have $F(\beta_n)(Y_{n-1}) = X_{n-1}$, so Y_{n-1} is indeed an element of $T^1_{X_{n-1}/A_{n-1}}$. Consider the following morphism of small extensions

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & k & \xrightarrow{\cdot \epsilon^{n+1}} & A_{n+1} & \xrightarrow{\alpha_n} & A_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi'_n} & & \\ e_2 : 0 & \longrightarrow & k & \xrightarrow{\cdot x^n y} & B_n & \xrightarrow{\gamma_n} & C_n & \longrightarrow & 0 \end{array}$$

where μ is the multiplication by $n+1$. From this we get the following diagram by applying the functor F:

$$\begin{array}{ccc} F(A_{n+1}) & \xrightarrow{F(\alpha_n)} & F(A_n) \\ \Big\downarrow{\scriptstyle F(\pi_n)} & & \Big\downarrow{\scriptstyle F(\pi'_n)} \\ F(B_n) & \xrightarrow{F(\gamma_n)} & F(C_n) \end{array}$$

The functor is supposed to satisfy (H5), therefore we can factor both $F(\gamma_n)$ and $F(\pi'_n)$ through $P_{n-1} := F(B_{n-1}) \times_{F(A_{n-1})} F(A_n)$ by a morphism $\tau := \tau_{B_{n-1},A_n,A_{n-1}}$ such that the following diagram commutes



Now consider $F(\pi'_n)(X_n) \in F(C_n)$. We see that $F(\pi'_n)(X_n)$ is in the image of $F(\gamma_n)$ iff

$$\tau\left(F(\pi_{n-1}) \times id_{F(A_n)}\right)(X_n) = \tau(Y_n, X_n)$$

is in the image of $F(\gamma_n)$ iff (Y_n, X_n) is in the image of $F(\xi_n) \times F(\beta_n)$ iff Y_n is in the image of

$$F(\xi_n)_{|T^1_{X_n/A_n}} : T^1_{X_n/A_n} \longrightarrow T^1_{X_{n-1}/A_{n-1}} \subset F(B_{n-1})$$

On the other hand, the morphism of small obstructions is compatible with the obstruction theories. So $F(\pi'_n)(X_n)$ can be lifted to $F(B_n)$ iff

$$v_F(e_2)(F(\pi'_n)(X_n)) = 0$$

which by compatibility is equivalent to $(Id_V \otimes \mu)(v_F(e_1)(X_n))(X_n) = 0$. $Id_V \otimes \mu$ is an isomorphism because $char(k) = 0$, so we obtain that this is the same as the vanishing of $v_F(e_1)(X_n)$ which in turn is equivalent to the existence of a lift $X_{n+1} \in F(A_{n+1})$. □

To use the T^1-lifting criterion, we need to check the following simple fact.

Lemma A.21. *Let F be a functor such that the natural restriction map $F(A_{n+1}) \to F(A_n)$ is surjective. Then F is smooth.*

Proof. In [FM98] there is a general proof using the factorization theorem (theorem 6.2 and corollary 6.4). However, for functors with finite-dimensional tangent spaces the situation is of course much simpler. Therefore, suppose that F has a hull X. Let $\mathcal{O}_X = k[[x_1, \dots, x_m]]/I$. If X is not smooth, then there is an infinitesimal curve $\mathcal{O}_X \to A_n$ which can not be extended to a curve $\mathcal{O}_X \to A_{n+1}$. This violates the surjectivity of $F(A_{n+1}) \to F(A_n)$. □

For our purpose, we need to know that the converse of the T^1-lifting theorem is true.

Lemma A.22. *Let $F \in Fun$ be smooth. Then the T^1-lifting property holds for F.*

Proof. Let $X_n \in F(A_n)$ be given. Then the element $F(\pi_{n-1})(X_n) \in T^1_{X_{n-1}/A_{n-1}}$ extends to $T^1_{X_n/A_n}$ because $F(\pi'_{n-1})(X_n) \in F(C_n)$ extends over B_n (F is smooth) and this extension obviously lies in $T^1_{X_n/A_n}$. □

A.2 Examples of controlling dg-Lie algebras

This part gives concrete examples which are applications of the general principle that "a deformation problem is governed by a dg-Lie algebra". All of these examples are of interest in their own right, but some of them (deformation of Lie algebras, the cotangent complex) are directly related to deformations of lagrangian singularities which are discussed in the second chapter. As an additional reference, we have used [Ste03].

A.2.1 The Kodaira-Spencer algebra

The Kodaira-Spencer algebra is the most classical example of a dg-Lie algebra controlling a deformation problem. Consider a complex manifold M, that is, a C^∞- manifold together with an integrable complex structure

$$J : TM \longrightarrow TM$$

The functor of deformations of M, that is, smooth families $M_S \to S$ of complex manifolds M_s with $M_0 = M$ reduces by the *Ehresmann* lemma to the functor of deformations of the complex structure. Now consider the dg-Lie algebra $(L, d, [\ ,\ ])$ with:

$$L^i := \Gamma(M, \mathcal{A}_M^{0,i} \otimes \Theta_M)$$

where $\mathcal{A}_M^{0,i}$ is the sheaf of C^∞-sections of the bundle of anti-holomorphic exterior forms of degree i. The differential d is induced from the Dolbeault differential $\overline{\partial}$ on antiholomorphic forms whereas the bracket comes from the Lie bracket on vector fields and from the exterior product on forms, explicitly:

$$[\Phi d\overline{z}_I, \Psi d\overline{z}_J] := [\Phi, \Psi] d\overline{z}_I \wedge d\overline{z}_J$$

Denote by Def_X the functor of deformations of the complex structure. Then we have the following statement:

Theorem A.23. *The functors Def_X and Def_L are equivalent.*

Proof. We will associate to an element of $MC_L(A)$ a deformation of the complex structure over $Spec(A)$. By definition, if $\gamma \in MC_L(A)$ then it is of the form

$$\gamma \in \Gamma(X, \mathcal{A}^{0,1} \otimes \Theta_M) \otimes \mathbf{m_A} = Hom_{C_M^\infty}(\overline{\Theta}_M, \Theta_M) \otimes \mathbf{m_A}$$

(where $\overline{\Theta}_M$ is the antiholomorphic tangent bundle). The graph of such a γ defines a deformed almost complex structure and it can be checked that this structure is integrable precisely iff $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$. On the other hand, the Lie algebra of the automorphism group of X is known to be the space of global holomorphic vector fields, which implies that $G_L(A) = \exp(L^0 \otimes \mathbf{m}_A)$ as required. $\square$

Corollary A.24. *The space of infinitesimal deformations of the complex structure is* $H^1(X, \Theta_X)$ *whereas* $T^2_{Def_X} = H^2(X, \Theta_X)$.

As an application, we prove that deformations of Calabi-Yau manifolds are unobstructed.

Corollary A.25. *Let* X *be a three dimensional Calabi-Yau manifold, that is, a compact Kähler manifold with* $c_1(X) = 0$. *Suppose moreover that* $H^1(X, \mathcal{O}_X) = 0$. *Then the functor* $Def(X)$ *is smooth.*

Proof. We follow [Nam94]. The vanishing of the first chern class is equivalent to the fact that the canonical bundle ω_X is trivial. By Serre duality, we then have

$$H^1(X, \Theta_X) \cong Hom_{\mathbb{C}}(H^2(X, \Omega^1_X), \mathbb{C})$$

The last lemma shows that this space equals $T^1_{Def(X)}$. We want to apply the T^1-lifting criterion, that is, we are going to show that for a given family $X_n \to A_n$, the restriction morphism

$$Hom_{A_n}\left(H^2(X_n, \Omega_{X_n/A_n}), A_n\right) \longrightarrow$$

$$Hom_{A_{n-1}}\left(H^2(X_{n-1}, \Omega_{X_{n-1}/A_{n-1}}), A_{n-1}\right)$$

is surjective. We will prove this in a number of steps. In fact, it will be sufficient to show that $H^2(\Omega^1_{X_n/A_n})$ is free over A_n and that $H^2(\Omega^1_{X_n/A_n}) \to H^2(\Omega^1_{X_{n-1}/A_{n-1}})$ is surjective for all $n > 0$. Then required surjectivity on the "Hom"-spaces follows by applying the functor $Hom_{A_n}\left(H^2(\Omega^1_{X_n/A_n}), -\right)$ (which is exact due the freeness of the module $H^2(\Omega^1_{X_n/A_n})$) to the exact sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{\cdot\epsilon^n} A_n \longrightarrow A_{n-1} \longrightarrow 0$$

Let us first prove that $H^2(\Omega^1_{X_n/A_n})$ is free over A_n. It suffices to show that the morphism $H^2(\Omega^1_{X_n/A_n}) \to H^2(\Omega^1_{X_{n+1}/A_{n+1}})$, given by multiplication by ϵ, is injective. The cohomology sequence of the short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_{X_n/A_n} \xrightarrow{\cdot\epsilon} \Omega^\bullet_{X_{n+1}/A_{n+1}} \longrightarrow \Omega^\bullet_X \longrightarrow 0$$

(this sequence is exact due to the smoothness of X) shows that it is sufficient to show the surjectivity of $H^1(\Omega^1_{X_{n+1}/A_{n+1}}) \to H^1(\Omega^1_X)$. In order to do that, one considers the map

$$dlog : H^1(\mathcal{O}^*_X) \longrightarrow H^1(\Omega^1_X)$$

of logarithmic differentiation. By Serre duality, $H^1(\mathcal{O}_X) = 0$ implies that $H^2(\mathcal{O}_X) = 0$, hence the map $H^1(\mathcal{O}^*_X) \to H^2(X, \mathbb{Z})$ is surjective. But again, $H^{1,0} = \overline{H^{0,1}} = 0$ so $H^1(\mathcal{O}^*_X) \otimes \mathbb{C} \to H^{1,1}$ is also surjective. This implies that the image of *dlog* generates $H^1(\Omega^1_X)$ as a $\mathbb{C}$-vector space. Now consider the diagram

$$\begin{array}{ccc} H^1(\mathcal{O}^*_{X_{n+1}}) & \longrightarrow & H^1(\mathcal{O}^*_X) \\ \downarrow{\scriptstyle dlog} & & \downarrow{\scriptstyle dlog} \\ H^1(\Omega^1_{X_{n+1}/A_{n+1}}) & \longrightarrow & H^1(\Omega^1_X) \end{array}$$

Take any class in $H^1(\Omega^1_X)$. We can write it as a $\mathbb{C}$-linear combination of elements in the image of *dlog*. Take the inverse image of these generators in $H^1(\mathcal{O}^*_X)$. If the map $H^1(\mathcal{O}^*_{X_{n+1}}) \to H^1(\mathcal{O}^*_X)$ is surjective, then we can find a preimage of the given class in $H^1(\Omega^1_{X_{n+1}/A_{n+1}})$. But surjectivity of $H^1(\mathcal{O}^*_{X_{n+1}}) \to H^1(\mathcal{O}^*_X)$ is clear: we are again left to show that multiplication by ϵ is injective as a map $H^2(\mathcal{O}^*_{X_n}) \to H^2(\mathcal{O}^*_{X_{n+1}})$. But from $H^2(\mathcal{O}_X) = 0$ we get that $H^2(\mathcal{O}^*_{X_k})$ injects in in $H^3(X_k, \mathbb{Z}_{X_k})$ (for any k) which is topological, i.e., the multiplication by ϵ is an injective map $H^3(X_n, \mathbb{Z}_{X_n}) \to H^3(X_{n+1}, \mathbb{Z}_{X_{n+1}})$.

It remains to show the surjectivity of

$$H^2(\Omega^1_{X_n/A_n}) \to H^2(\Omega^1_{X_{n-1}/A_{n-1}})$$

This is much easier. In fact, as before we get from the long exact cohomology sequence that it is sufficient to prove $H^3(\Omega^1_X) = 0$. By using duality once again we have $H^3(\Omega^1_X) = Hom(\Omega^1_X, \mathcal{O}_X)' = H^0(X, \Theta_X)'$ where $'$ stands for the vector space dual. Interior derivation of the canonical three form gives an isomorphism $\Theta_X = \Omega^2$ so that

$$H^3(\Omega^1_X) = H^0(X, \Theta_X)' = H^0(X, \Omega^2_X) = \overline{H^2(X, \mathcal{O}_X)}$$

But $H^2(X, \mathcal{O}_X) = 0$ as we have already remarked. □

We make another remark on deformations of Calabi-Yau manifolds: There is a construction of a dg-Lie algebra (due to Kontsevich and Barannikov, see [BK98]), canonically associated to any Calabi-Yau which includes the Kodaira-Spencer dg-Lie algebra. Its definition is rather simple, one considers the exterior algebra of the tangent sheaf and the defines the graded space

$$L^i := \Gamma(M, \mathcal{A}_M^{0,p-i+1} \otimes \bigwedge^p \Theta_M)$$

together with the Dolbeault differential $\overline{\partial}$ as above. The bracket is induced from the the product on forms and from the so-called *Schouten-Nijenhuis*-bracket on polyvector fields. One can show that the versal deformation space (in the formal sense) is the total cohomology space

$$\mathbf{H} := \oplus_{i=1}^n H^i(X, \mathbb{C})$$

of the manifold X. This dg-Lie algebra parameterizes therefore a more general object attached to X than just its complex structure. Apparently, this object is the derived category of coherent sheaves on X, viewed as an A_∞-category. Moreover, there is some additional structure on L, formalized as the so-called dGBV-algebra (differential Gerstenhaber-Batalin-Vilkovisky algebra) which equips the versal deformation space H with the structure of a (formal) Frobenius manifold. This structure has become very important to study the *mirror symmetry* phenomenon, i.e., to identify Calabi-Yau manifolds from apparently very different origins.

A.2.2 Deformation of associative, commutative and Lie algebras

This section deals with deformation of purely algebraic structures: associative, commutative and Lie algebras . The corresponding differential graded Lie algebras are constructed quite similarly. The material in this section is rather classical, a standard reference is [GS88].

We start with an associative algebra A over a field k. A is seen as a vector space over k together with a k-bilinear multiplication

$$\mu : A \times A \longrightarrow A$$

such that (associativity condition) $\mu\,(a, \mu(b, c)) = \mu\,(\mu(a, b), c)$. A deformation is a family (over a base S) of maps $\mu_t : A \times A \longrightarrow A$ where t is a parameter from the base. As we want to deal with arbitrary bases (e.g., artinien rings), we define more carefully the functor $Def_A(S)$ to be an associative S-algebra structure on $A \otimes_k S$ modulo isomorphisms. We will now construct a dg-Lie algebra controlling this deformation problem.

Consider first a slightly more general situation. Let M be an A-bimodule (where the bimodule structure is given by morphisms $\alpha : A \times M \to M$ and $\beta : M \times A \to M$) and

$$C^n(A, M) := Hom_K(A^{\otimes n}, M)$$

be the vector space of k-multilinear maps from $A \times \ldots \times A$ to M. Define a differential $\delta : C^n(A, M) \to C^{n+1}(A, M)$ by

$$\begin{aligned} &\delta(\phi)(a_0 \otimes \ldots \otimes a_n) := \alpha\,(a, \phi(a_1, \ldots, a_n)) \\ &+ \textstyle\sum_{i=1}^{n} (-1)^i \phi(a_0, \ldots, \mu(a_{i-1}, a_i), \ldots, a_n) \\ &+ (-1)^{n+1} \beta\,(\phi(a_0, \ldots, a_{n-1}), a_n) \end{aligned}$$

One has to check that δ is indeed a differential. The resulting cohomology $H^k(A, M)$ of this complex is called the *Hochschild*-cohomology of the algebra A with coefficients in M.

Now consider extensions

$$e : 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

of the algebra A by an A-bimodule M such that B is a k-algebra (with multiplication μ_e) and the map $B \to A$ is a map of k-algebras. Moreover, we require that the two B-bimodule structures of M (the one given by the inclusion $M \hookrightarrow B$ and the one given by the algebra map $B \to A$) coincide. This immediately implies that M is a two-sided ideal in B with $M^2 = 0$. Two extensions are called equivalent iff there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

The set of isomorphism classes of such extensions forms a vector space by the usual *Baer sum*, where the zero element consists of the semi-direct product $B = A \times M$ with multiplication

$$\mu_0\left((a,m),(a',m')\right) = \left(\mu(a,a'), \alpha(a,m') + \beta(m,a')\right)$$

Lemma A.26. *This vector space is isomorphic to $H^2(A,M)$. In particular, extensions of A by itself modulo isomorphisms are classified by $H^2(A,A)$.*

Proof. For any extension e, the algebra B is isomorphic to $A \times M$ as a k-vector space. Then the first component of the multiplication μ_e is equal to μ, because B/M is isomorphic to A *as an algebra.* On the other hand we know that

$$\begin{aligned} \mu_e(a,m') &= \alpha(a,m') \\ \mu_e(m,a') &= \beta(m,a') \end{aligned}$$

Finally, $\mu_e(m,m') = 0$, therefore, the multiplication is given by

$$\mu_e\left((a,m),(a',m')\right) = \left(\mu(a,a'), \alpha(a,m') + \beta(m,a') + \lambda(a,a')\right)$$

for some $\lambda \in C^2(A,M)$. The associativity equation for B reads:

$$\mu_e\left(\mu_e\left((a_1,m_1),(a_2,m_2)\right),(a_3,m_3)\right) =$$

$$\mu_e\left((a_1,m_1),\mu_e\left((a_2,m_2),(a_3,m_3)\right)\right)$$

which is equivalent to

$$\begin{gathered} \alpha\left(a_1,\alpha(a_2,m_3)\right) + \alpha\left(a_1,\beta(m_2,a_3)\right) + \alpha\left(a_1,\lambda(a_2,a_3)\right) + \\ \beta\left(m_1,\mu(a_2,a_3)\right) + \lambda\left(a_1,\mu(a_2,a_3)\right) \\ = \\ \alpha\left(\mu(a_1,a_2),m_3\right) + \beta\left(\alpha(a_1,m_2),a_3\right) + \beta\left(\lambda(a_1,a_2),a_3\right) + \\ \beta\left(\beta(m_1,a_2),a_3\right) + \lambda\left(\mu(a_1,a_2),a_3\right) \end{gathered}$$

By definition, we have

$$\begin{aligned} \alpha\left(a_1,\alpha(a_2,m_3)\right) &= \alpha\left(\mu(a_1,a_2),m_3\right) \\ \alpha\left(a_1,\beta(m_2,a_3)\right) &= \beta\left(\alpha(a_1,m_2),a_3\right) \\ \beta\left(m_1,\mu(a_2,a_3)\right) &= \beta\left(\beta(m_1,a_2),a_3\right) \end{aligned}$$

Thus associativity is equivalent to

$$\alpha\left(a_1, \lambda(a_2,a_3)\right) + \lambda\left(a_1, \mu(a_2,a_3)\right) = \beta\left(\beta(m_1,a_2),a_3\right) + \lambda\left(\mu(a_1,a_2),a_3\right)$$

meaning that $\delta\lambda = 0$.

Now consider an extension e which is equivalent to e_0 by a commutative diagram as above. The arrow $g : B \to A \times M$ (where the latter algebra corresponds to e_0) is necessarily an isomorphism and of the form $g(a,m) = (a, m + h(a))$ for some $h \in C^1(A,M)$ (this follows immediately from the commutativity). Its inverse is given by $g^{-1}(a,m) = (a, m - h(a))$. To say that e and e_0 are equivalent is to say that g is an algebra isomorphism, i.e.:

$$g\left(\mu_e\left((a,m),(\widetilde{a},\widetilde{m})\right)\right) = \mu_{e_0}\left(g(\widetilde{a},\widetilde{m}), g(\widetilde{a},\widetilde{m})\right)$$

that is:

$$\begin{aligned}
\mu_e\left((a,m),(\widetilde{a},\widetilde{m})\right) &= g^{-1}\left(\mu_{e_0}\left(g(a,m), g(\widetilde{a},\widetilde{m})\right)\right) \\
&= g^{-1}\left(\mu_{e_0}\left((a, m+h(a)),(\widetilde{a},\widetilde{m}+h(\widetilde{a}))\right)\right) \\
&= g^{-1}\left(\mu(a,\widetilde{a}), \alpha(a,\widetilde{m}+h(\widetilde{a})) + \beta(m+h(a),\widetilde{a})\right)
\end{aligned}$$

Therefore we get

$$\alpha(a,\widetilde{m}) + \beta(m,\widetilde{a}) + \lambda(a,\widetilde{a}) = \alpha(a,\widetilde{m}+h(\widetilde{a})) + \beta(m+h(a),\widetilde{a}) - h(\mu(a,\widetilde{a}))$$

and thus

$$\lambda(a,\widetilde{a}) = \alpha(a,h(\widetilde{a})) + \beta(h(a),\widetilde{a}) - h(\mu(a,\widetilde{a}))$$

Therefore $\lambda = \delta h$. This finishes the proof of the lemma. □

It is clear that infinitesimal deformations of the algebra A, that is, $k[\epsilon]/\epsilon^2$-algebra structures on $A[\epsilon]/\epsilon^2$ are precisely extensions of A by itself. Therefore, the tangent space of the functor Def_A is isomorphic to $H^2(A,A)$. Thus we have to construct the structure of a dg-Lie algebra on the Hochschild complex $C^\bullet(A,A)$. In order to define the Lie bracket, we first shift (somewhat artificially) the degree of the terms of this complex by setting $\overline{C}^n(A,M) := C^{n+1}(A,M)$. Then we define the *composition product*

$$\begin{aligned}
\overline{C}^n(A,A) \times \overline{C}^m(A,A) &\longrightarrow \overline{C}^{n+m}(A,A) \\
(g,f) &\longmapsto g \circ f
\end{aligned}$$

with

$$(g \circ f)(a_1, \ldots, a_{n+m+1}) :=$$

$$\sum_{i=1}^{n+1} (-1)^{m(i-1)} g(a_1, \ldots, a_{i-1}, f(a_i, \ldots, a_{i+m}), a_{i+m+1}, \ldots, a_{n+m+1})$$

The bracket is just the commutator with respect to this product:

$$\begin{array}{rcl} [\ ,\ ] : \overline{C}^p \times \overline{C}^p & \longrightarrow & \overline{C}^{p+q} \\ (g, f) & \longmapsto & g \circ f - (-1)^{pq} f \circ g \end{array}$$

Theorem A.27. *The triple* $(\overline{C}^\bullet(A, A), \delta, [\ ,\ ])$ *is a dg-Lie algebra. Moreover the associated functor* $Def_{\overline{C}}$ *is isomorphic to* Def_A.

Proof. To prove the first statement, three things have to be checked: the anti-commutativity and Jacobi identity of the bracket and the compatibility between bracket and differential (all three statements have to been understand in the graded sense). We first remark that the differential can be written in terms of the bracket as

$$\delta\phi = (-1)^{deg(\phi)+1} \phi \circ \mu - \mu \circ \phi = -[\mu, \phi]$$

for any $\phi \in \overline{C}^\bullet(A, A)$ (note that we use shifted degrees here). Then the equality (compatibility of bracket and differential)

$$\delta[\phi, \psi] = [\delta\phi, \psi] + (-1)^{deg(\phi)} [\phi, \delta\psi]$$

is equivalent to the graded Jacobi identity. To prove it (and the anti-commutativity), one has to check explicitly rather huge identities for the product $\circ$. We refrain from doing this here.

Now consider a ring $S \in$ **Art**. To any element $\lambda \in MC_{\overline{C}}(S)$ we associate the "deformed multiplication"

$$\mu_\lambda := \mu + \lambda : (A \otimes_k S) \otimes_S (A \otimes_k S) \to A \otimes_k S$$

This obviously defines an algebra structure (over S) on $A \otimes_k S$. We want to know whether it is associative, this means by definition of the composition product:

$$(\mu_\lambda \circ \mu_\lambda)(a, b, c) = \mu_\lambda(\mu_\lambda(a, b), c) - \mu_\lambda(a, \mu_\lambda(b, c)) = 0$$

So the deformed multiplication is associative iff

$$(\mu+\lambda)\circ(\mu+\lambda)=\mu\circ\mu+\mu\circ\lambda+\lambda\circ\mu+\lambda\circ\lambda=0$$

The original multiplication was associative, therefore $\mu\circ\mu=0$. Moreover, $deg(\lambda)=1$ so $[\lambda,\lambda]=2\lambda\circ\lambda$. Therefore the associativity condition for μ_λ is equivalent to

$$\delta\lambda+\frac{1}{2}[\lambda,\lambda]=0$$

This means that we have a surjective morphism of functors $MC_{\overline{C}}\to Def_A$. Now it can be verified that whenever a given deformation μ_λ over S is altered by an automorphism from $\exp(C^1)$, then the resulting deformation can be transformed back by an automorphism of $A\otimes S$. Moreover, all automorphisms of $A\otimes S$ are of this type, therefore, the induced morphism of functors $Def_{\overline{C}}\to Def_A$ is an isomorphism. □

The cases of deformation of commutative and Lie algebras can now be describe rather briefly. Let A be an associative and commutative algebra. Then we want to consider commutative deformations, consequently, we look for a dg-Lie algebra which is a subcomplex of the Hochschild complex. Consider the symmetric group $\mathcal{S}_n$ and define for all $0<r<n$ a pure r-**shuffle** to be a permutation $\pi\in\mathcal{S}_n$ such that $\pi(1)<\ldots<\pi(r)$ and $\pi(r+1)<\ldots<\pi(n)$. Then the r-th shuffle operator is $s_r:=\sum_{\text{pure shuffles}}sgn(\pi)\pi$. Now we define the n-th Harrison cochain module to be

$$Ch^n(A,M):=\{\phi\in C^n(A,M)\mid\phi\circ s_r=0\quad\forall r\}$$

Theorem A.28. *$Ch^\bullet(A,M)$ together with the Hochschild differential is a subcomplex of $C^\bullet(A,M)$. Moreover, for $M=A$, the bracket from the Hochschild complex restricts to $Ch^\bullet(A,A)$, which therefore becomes a sub dg-Lie algebra of $C^n(A,A)$. The associated functor $Def_{\overline{Ch}}$ is the functor of **commutative** deformations of A.*

Proof. This is proved with the same methods as in the associative case. We only remark that for $n=0$ and $n=1$ there are no shuffles so Hochschild and Harrison cohomology coincide. On the other hand, for $n=2$ we have precisely one shuffle, namely $a\otimes b-b\otimes a$, therefore,

$H^2(Ch^\bullet(A,M))$ classifies commutative extensions of A by a symmetric A-bimodule M. In particular, $H^2(Ch^\bullet(A,A))$ are the infinitesimal commutative deformations of A. □

Finally, we consider deformations of Lie algebras. We give only the definition of the corresponding dg-Lie algebra, referring to [GS88] for details. Let $\mathfrak{g}$ be a Lie algebra over k and M be an $\mathfrak{g}$-module (which is by definition a module over the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$). Then we define the module

$$C^n(A,M) := Hom\left(\bigwedge^n \mathfrak{g}, M\right)$$

and a differential $\delta : C^n(\mathfrak{g},M) \to C^{n+1}(\mathfrak{g},M)$ by

$$\begin{aligned}&(\delta\phi)\left(g_1 \wedge \ldots \wedge g_{n+1}\right) := \\ &\textstyle\sum_{i=1}^{n+1} (-1)^i \left[g_i, \phi\left(g_1 \wedge \ldots \wedge \widehat{g}_i \wedge \ldots g_{n+1}\right)\right] \\ &+ \sum_{1\leq i<j\leq n+1} (-1)^{i+j-1} \phi\left([g_i,g_j] \wedge g_1 \wedge \ldots \wedge \widehat{g}_i \wedge \ldots \wedge \widehat{g}_j \wedge \ldots \wedge g_{n+1}\right)\end{aligned}$$

In the case $M = \mathfrak{g}$ there is a bracket, defined for two elements $\phi \in C^n(\mathfrak{g},\mathfrak{g})$ and $\psi \in C^m(\mathfrak{g},\mathfrak{g})$ as

$$[\phi,\psi] = \phi \wedge \psi - (-1)^{(m-1)(n-1)} \psi \wedge \phi$$

where

$$(\phi \wedge \psi)(g_1, \ldots, g_{n+m-1}) =$$

$$\sum_{\text{pure shuffles}} sgn(\pi)\phi\left(\psi(a_{\pi(1)}, \ldots, a_{\pi(n)}), a_{\pi(n+1)}, \ldots, a_{\pi(m+n-1)}\right)$$

Theorem A.29. $(\overline{C}^\bullet(A,A), \delta, [\ ,\ ])$ *(reduced degree) is the controlling dg-Lie algebra of the Lie algebra deformation problem.*

A.2.3 The cotangent complex

We will construct a dg-Lie algebra which controls flat deformations of singularities. Here we consider only germs of complex spaces and their deformations. The global case is considerably more involved as one has to take into account deformations of singularities and of complex structures simultaneously (see, e.g., [BM97]). Our main reference for this section is [Man01a]. Consider an analytic algebra A, given as a quotient

$$A_0 \longrightarrow A := A_0/(f_1, \dots, f_k)$$

where $A_0 := \mathbb{K}\{x_1, \dots, x_n\}$. We first construct a *resolvent* of A, which is by definition a free differential graded A_0-algebra R, concentrated in negative degrees, with finitely many generators in each degree, which is quasi-isomorphic to A. The idea of the construction is rather simple. We will define a chain

$$R(0) := A_0 \subset R(1) \subset R(2) \subset \dots$$

of DGA's of the above type, not necessarily acyclic but where in each step some of the remaining cohomology has been killed. Then the union

$$R := \bigcup_{i=0}^{\infty} R(i)$$

will be quasi-isomorphic to A.

Define $R(0)$ to be the single degree complex A_0 concentrated in degree zero. Then set

$$R(1) := \mathbb{K}\{x_1, \dots, x_n\}[y_1, \dots, y_{s_1}]$$

with $s_1 := k$ and $deg(x_i) = 0$ and $deg(y_j) = -1$. The differential δ is uniquely determined by

$$\delta(x_i) = 0 \quad \text{and} \quad \delta(y_j) = f_j$$

and by requiring that $R(1)$ is a DGA. Now we proceed inductively. Suppose that $R(i)$ is constructed such that $H^j(R(i)) = 0$ for all $j > -i$. Then choose a system $h_1^{(i)}, \dots, h_{t_i}^{(i)}$ of generators of $H^{-i}(R(i))$ and set

$$R(i+1) := R(i)[y_{s_i+1}, \dots, y_{s_{i+1}}]$$

with $s_{i+1} := s_i + t_i$, $deg(y_l) = -i - 1$ and $\delta(y_l) = h_{l-s_i}^{(i)}$ for $l \in \{s_i + 1, \dots, s_{i+1}\}$.

Now for any DGA $(O, d, \bullet)$ over $\mathbb{K}$, we consider the set $Der_{\mathbb{K}}(O, O)$ of all derivations of O into itself. More precisely:

$$\begin{aligned} Der^n_{\mathbb{K}}(O, O) \quad &:= \quad \{\Phi \in Hom_{\mathbb{K}}(O, O) \mid \Phi(O_k) \subset O_{n+k}, \ , \Phi(\mathbb{K}) = 0, \\ &\qquad \Phi(a \bullet b) = \Phi(a) \bullet b + (-1)^{n \cdot deg(a)} a \bullet \Phi(b)\} \\ Der_{\mathbb{K}}(O, O) \quad &:= \quad \bigoplus_{n \in \mathbb{Z}} Der^n_{\mathbb{K}}(O, O) \end{aligned}$$

This definition makes $Der_{\mathbb{K}}(O, O)$ into a dg-Lie algebra, where the (graded) bracket is the commutator of derivations and the differential d is defined as the commutator with δ. One can show that this construction is unique up to homotopy equivalence.

Definition A.30. *Let A be an analytic algebra as above. Define*

$$(L_A, d, [\ ,\ ]) := Der_{\mathbb{K}}(R, R)$$

to be the dg-Lie algebra of derivations of the resolvent of A.

The importance of this construction is given by the following theorem.

Theorem A.31. *Denote by Def_A the functor of flat deformations of the analytic algebra A. Then we have an isomorphism of functors $Def_{L_A} \to Def_A$.*

Proof. First we define a transformation

$$MC_{L_A} \longrightarrow Def_A$$

So let B be an Artin ring and take an element $\eta \in L^1_A \otimes \mathbf{m}_B = Der^1_{\mathbb{K}}(R, R) \otimes \mathbf{m}_B$. Then we can consider the "perturbed" differential

$$\delta_\eta := \delta + \eta : R_i \otimes B \longrightarrow R_{i+1} \otimes B$$

Let us calculate its square: As $deg(\delta_\eta) = 1$, we see that $[\delta_\eta, \delta_\eta] = 2\delta^2_\eta$ and so

$$2\delta^2_\eta = [\delta + \eta, \delta + \eta] = \delta^2 + [\delta, \eta] + [\eta, \delta] + [\eta, \eta] = 2[\delta, \eta] + [\eta, \eta] = 2d\eta + [\eta, \eta]$$

This implies that δ_η is a differential iff $d\eta + \frac{1}{2}[\eta, \eta] = 0$, i.e., iff $\eta \in MC_{L_A}(B)$. It is well-known in homological algebra that a complex of modules flat over B is exact iff its reduction modulo $\mathbf{m}_B$ is exact. Therefore, $R_B := (R \otimes B, \delta + \eta)$ is a resolution of

$$A_B := coker\left(R_{-1} \otimes B \xrightarrow{\delta_\eta} R_0 \otimes B\right)$$

As $R_B \otimes_B \mathbb{K} = R$, we see that A_B is a family over B with special fibre isomorphic to A. It remains to show that $A_B \to B$ is flat. But $Tor_1^B(A_B, \mathbb{K}) = H^{-1}(R) = 0$, so we are done by the local flatness criterion. This shows that we have defined a morphism $MC_{L_A} \longrightarrow Def_A$ by sending $\eta \in MC_{L_A}(B)$ to the isomorphism class of A_B.

As a second step, we now prove that this morphism is surjective. So let us be given a flat family A_B which specializes to the algebra A over $\mathbb{K}$. We have the morphism $R_0 \otimes B \to R_0 \to A$, and the surjection $A_B \to A$. R_0 was a free $\mathbb{K}$-algebra, so this yields a morphism of flat B-algebras $R_0 \otimes B \to A_B$. As its reduction over the special point is surjective, the morphism is itself surjective. So the situation is as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{B,0} & \longrightarrow & R_0 \otimes B & \longrightarrow & A_B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_0 & \longrightarrow & R_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where $I_{B,0}$ is flat over B. Therefore we can extend the differential δ on $R(1)$ to a differential δ_B on $R(1) \otimes B$ by choosing lifts F_i of the elements $\delta(y_i) = f_i \in I_0$ to $I_{B,0}$ and setting $\delta_B(y_i) = F_i$. Remark that now we have $H^0(R(1) \otimes B) = 0$. Then we proceed inductively: at each step k flatness over A of the kernel of ∂_B at degree k guarantees the existence of an extension of the given differential on R.

So we obtain a DGA $(R \otimes B, \delta_B, \bullet)$ which is quasi-isomorphic to A_B and whose restriction over $\mathbb{K}$ is the given resolvent of A. But this also implies that the differential can be written as $\delta_B = \delta + \eta$ with $\eta \in \mathbf{m}_B$. Therefore, we get an $\eta \in MC_{L_A}(B)$ which shows that the above transformation $MC_{L_A} \to Def_A$ is surjective.

Remark that given $\xi \otimes b \in G_{L_A}(B) = Der^0_{\mathbb{K}}(R, R) \otimes \mathbf{m}_B$, we get an automorphism

$$\begin{array}{rcl} e^{\xi\otimes b}: R\otimes B & \longrightarrow & R\otimes B \\ x\otimes \widetilde{b} & \longmapsto & \sum_{i=0}^{\infty}\frac{1}{i!}\xi^i(x)\otimes b^i\widetilde{b} \end{array}$$

which induces the identity on R and sends the differential $\delta+\eta$ to $\delta+e^{\xi\otimes b}(\eta)$. In particular, we have

$$\begin{array}{l} e^{\xi\otimes b}\left(coker(\delta+\eta: R_{-1}\otimes B\to R_0\otimes B)\right) \\ = coker\left(\delta+e^{\xi\otimes b}(\eta): R_{-1}\otimes B\to R_0\otimes B\right) \end{array}$$

This means that the morphism $MC_{L_A}\to Def_A$ factors through $MC_{L_A}\to Def_{L_A}\to Def_A$ and obviously, $Def_{L_A}\to Def_A$ is surjective. The last step is now to show that $Def_{L_A}\to Def_A$ is also injective. So take $\eta,\eta'\in MC_{L_A}(B)$ and consider the two complexes $(R\otimes B,\delta+\eta)$ and $(R\otimes B,\delta+\eta')$. We suppose that the induced deformations A_B and A'_B are isomorphic. It can be proved that this isomorphism can be lifted to an automorphism $g_0: R_0\otimes B\to R_0\otimes B$, so that

$$g_0\left((\delta+\eta)(R_{-1}\otimes B)\right)\cong(\delta+\eta')(R_{-1}\otimes B)$$

Moreover, g restricts to the identity over $\mathbb{K}$. This extends to an automorphism $g: R\otimes B\to R\otimes B$, such that $g\circ(\delta+\eta)=\delta+\eta'$ and even $\delta+g\circ\eta=\delta+\eta'$ as $g(\delta)=\delta$. But every automorphism of $R\otimes B$ is the exponential of a nilpotent derivation of degree zero, so there is $l\in Der^0_{\mathbb{K}}(R,R)\otimes\mathbf{m}_B$ with $e^l=g$. Then we have $e^l(\eta)=\eta'$ and this means that the classes of η and η' in Def_{L_A} are equal. This finishes the proof. □

Corollary A.32. *The spaces of infinitesimal automorphisms, infinitesimal deformations and obstructions of an analytic algebra $A:=A_0/I$ with $I=(f_1,\dots,f_k)$, denoted by T^0_A, T^1_A and T^2_A, respectively, are as follows:*

1. $T^0_A=Hom_A(\Omega^1_A,A)=:\Theta_{A/\mathbb{K}}$
2. $T^1_A=coker\left(\Theta_{A_0/\mathbb{K}}\to Hom_{A_0}(I,A)\right)$
3. $T^2_A=coker\left(Hom_{A_0}(R_{-1},A)\to Hom_{A_0}(\mathcal{R},A)\right)$, *where $\mathcal{R}$ is the module of relations of I.*

Moreover, the primary obstruction map can be described as follows: Let $\phi \in Hom_{A_0}(I, A)$ be a first-order deformation. Then define an element in $Hom_{A_0}(\mathcal{R}, A)$ by sending a relation $r_1, \ldots, r_k$ between the generators of I to the sum $\sum_{i=0}^{k} s_i \cdot \phi(f_i)$. Here $s_1, \ldots, s_k$ is a lifting of the relation $r_1, \ldots, r_k$, i.e., $\sum_{i=0}^{k}(f_i + \epsilon\phi(f_i)(r_i + \epsilon s_i) \in I$ (The existence of such a lifting is guaranteed by the flatness of the given deformation).

Proof. We have to calculate the cohomology of the dg-Lie algebra L. We use the following modification of L: Let R be the resolvent of the algebra A constructed above and consider $H := Der_{R_0}(R, R)$. This also has the structure of a dg-Lie algebra and there is an exact sequence of complexes

$$0 \longrightarrow H \longrightarrow L \longrightarrow Der_{\mathbb{K}}(R_0, R) \longrightarrow 0$$

Furthermore, we have

$$\begin{aligned} H^0(Der_{\mathbb{K}}(R_0, R)) &= \left\{\alpha \in Der^0_{\mathbb{K}}(R_0, R) \mid \delta \circ \alpha = \alpha \circ \delta\right\} \\ &= Der_{\mathbb{K}}(R_0, A) \end{aligned}$$

and $H^i(Der_{\mathbb{K}}(R_0, R)) = 0$ for $i \neq 0$ (because $Der_{\mathbb{K}}(R_0, R)$ is concentrated in degrees ≤ 0, R is exact in degree ≤ 0 and R_0 is free). Moreover, we have $H^i(H) = 0$ for $i \leq 0$ and therefore $H^i(L) = H^i(H)$ for $i > 1$. We get an exact sequence

$$0 \longrightarrow H^0(L) \longrightarrow Der_{\mathbb{K}}(R_0, A) \longrightarrow H^1(H) \longrightarrow H^1(L) \longrightarrow 0$$

Any class $\alpha \in H^0(L)$ induces in particular an $\alpha \in Der_{\mathbb{K}}(R_0, R_0)$ with $\alpha(I) \subset I$, therefore $\alpha \in Der_{\mathbb{K}}(A, A) = \Theta_{A/\mathbb{K}}$. On the other hand, given any $\beta \in Der_{\mathbb{K}}(A, A)$, we can extend it to a derivation of R because of the exactness of R in negative degree and get something in $H^0(L)$. Therefore, $H^0(L) = \Theta_{A/\mathbb{K}}$. Now consider a cocycle representing a class in $H^1(H)$, that is, an $\eta \in Der^1_{R_0}(R, R)$ with $\delta\eta = -\eta\delta$. In particular, η sends R_{-1} into R_0 and $\eta(\delta(R_{-2})) = \delta(\eta(R_{-2})) \subset \delta(R_{-1})$. So we get

$$\eta : R_{-1}/\delta(R_{-2}) \longrightarrow R_0/\delta(R_{-1})$$

but by the construction of the resolvent R we have $R_{-1}/\delta(R_{-2}) = I$ and $R_0/\delta(R_{-1}) = A$. So we obtain a well defined element in $Hom_{R_0}(I, A)$. One sees that η sends I into itself iff it is a coboundary. This means that we get a well-defined injective map $H^1(H) \to Hom_{R_0}(I, A)$. Surjectivity is obvious, because as above, a derivation from R_{-1} to R_0 coming from a morphism in $Hom_{R_0}(I, A)$ can be extended to the whole R. The above exact sequence thus reads

$$0 \longrightarrow \Theta_{A/\mathbb{K}} \longrightarrow \Theta_{R_0/\mathbb{K}} \longrightarrow Hom_{A_0}(I, A) \longrightarrow H^1(L) \longrightarrow 0$$

This proves the statement on T^1_A. Next we calculate $H^2(L) = H^2(H)$. First note that the module $\mathcal{R}$ of relations of I is canonically identified with the image of $\delta : R_{-2} \to R_{-1}$. Then given $\vartheta \in Der^2_{\mathbb{K}}(R, R)$ with $\delta\vartheta = \vartheta\delta$, define an element of $Hom_{A_0}(\mathcal{R}, A)$ by sending $r \in \mathcal{R}$ to the class of $\vartheta(\widehat{r})$ in A, where $\widehat{r}$ is a preimage of r in R_{-2}. This is well defined: if the chosen preimage is $\widehat{r}$ is in $\mathrm{Im}(\delta : R_{-3} \to R_{-2})$, i.e., $\widehat{r} = \delta(r')$, then $\vartheta(\widehat{r}) = \vartheta(\delta(r')) = 0 \in A$. Moreover, the defined morphism from $\mathrm{Im}(\delta : R_{-2} \to R_{-1})$ to A extends to R_{-2} iff $\vartheta = \eta \circ \delta + \delta \circ \eta$ for some $\eta \in Der^1_{\mathbb{K}}(R, R)$, i.e., iff ϑ is a coboundary. Therefore, we have a morphism

$$H^2(L) \longrightarrow coker\,(Hom_{A_0}(R_{-1}, A) \to Hom_{A_0}(\mathcal{R}, A))$$

which is easily seen to be an isomorphism. From the general discussion above (see lemma A.16 on page 149) we know that the primary obstruction map is given by

$$\begin{array}{rcl} ob : T^1_A & \longrightarrow & T^2_A \\ \phi & \longmapsto & \frac{1}{2}[\phi, \phi] \end{array}$$

Then given any relation $\mathbf{r} \in \mathcal{R}$ (which we see as an element of R_{-2}), we have to prove that the class of

$$\frac{1}{2}[\phi, \phi](\mathbf{r}) = (\phi \circ \phi)\,(\mathbf{r})$$

in A coincides with $\sum_{i=0}^k s_i \cdot \phi(f_i)$, where $(s_1, \ldots, s_k)$ is a lifting of the relation $\mathbf{r} = (r_1, \ldots, r_k)$. This is clear: Consider the perturbed differential $\delta_\phi = \delta + \epsilon\phi$, then

$$\delta_\phi(\mathbf{r}) = \sum_{i=1}^k (r_i y_i + \epsilon\phi(\mathbf{r})) = \sum_{i=1}^k (r_i + \epsilon s_i)\, y_i$$

where y_i are the generators of the free R_0-module R_{-1}. On the other hand, we have $\delta_\phi(y_i) = f_i + \epsilon\phi(f_i)$, so that, $(\phi \circ \phi)(\mathbf{r}) = \phi(\sum_{i=1}^k s_i y_i) = \sum_{i=1}^k s_i \phi(f_i)$. This finishes the proof. □

Appendix B

Algebraic analysis

Algebraic analysis, or in other words, the theory of (algebraic or analytic) $\mathcal{D}$-modules is the study of systems of differential equations by algebraic methods. More precisely, to any system of such equations on a, say, complex manifold X is associated a sheaf of modules over the sheaf of non-commutative rings of differential operators on X. Any such $\mathcal{D}_X$-module M possesses a *characteristic* variety $char(M)$, which is in some sense a differential analog of the usual support of an $\mathcal{O}_X$-module. Namely, it is an analytic subspace of the cotangent bundle T^*X with the crucial property that it is a *co-isotropic* subvariety with respect to the usual symplectic structure of T^*X. The special class of $\mathcal{D}_X$-modules for which it is lagrangian, i.e., $dim(char(M)) = dimX$ is called *holonomic* and is of particular importance. We will explain the notions mentioned here in more detail, in particular characteristic varieties. Good general references for $\mathcal{D}$-modules are [Pha79], [GM93]. See also the comprehensive monograph [Bjö93]. In this chapter we restrict our attention to the analytic $\mathcal{D}$-module theory over the complex numbers.

B.1 The characteristic variety

Let X be a complex analytic manifold. Let $(U; (x_1, \ldots, x_n)) \subset X$ be a coordinate chart. Then there exists the *ring of differential operators with* *holomorphic coefficient in* U, denoted $\mathcal{D}_X(U)$ and defined as follows:

$$\mathcal{D}_X(U) \quad := \quad \bigcup_{n=0}^{\infty} \mathcal{D}_X(U)(n)$$

$$\mathcal{D}_X(U)(n) \quad := \quad \left\{ P = \sum_{|I|=0}^{n} a_I \partial_I \mid a_I \in \mathcal{O}_X(U) \right\}$$

where $I = (i_1, \ldots, i_n)$ is a multi-index and $\partial_I := \partial_{x_{i_1}} \ldots \partial_{x_{i_n}}$. Here ∂_{x_i} is the $\mathbb{C}$-linear endomorphism of $\mathcal{O}_X(U)$ of differentiation with respect to x_i. Note that $\mathcal{D}_X(U)(0)$ is naturally equal to $\mathcal{O}_X(U)$ where a function on U is acting on $\mathcal{O}_X(U)$ by multiplication. Then the multiplication law in the ring $\mathcal{D}_X(U)$ is given by the usual commutator rules of differential operators, i.e.:

$$\partial_{x_i} x_j - x_j \partial_{x_i} = \partial_{ij}$$
$$\partial_{x_i} \partial_{x_j} - \partial_{x_j} \partial_{x_i} = 0$$

We note the following characterizations of $\mathcal{D}_X(U)$ and $\mathcal{D}_X(U)(n)$.

Lemma B.1. *Consider the ring* $End_{\mathbb{C}}(\mathcal{O}_X(U))$ *of* $\mathbb{C}$*-linear endomorphisms of* $\mathcal{O}_X(U)$.

- *The ring* $\mathcal{D}_X(U)$ *is isomorphic to the subring of* $End_{\mathbb{C}}(\mathcal{O}_X(U))$ *generated by* $\mathcal{O}_X(U)$ *and the operators* ∂_{x_i}.
- *We have*

$$\mathcal{D}_X(U)(n) \quad = \quad \left\{ \begin{array}{l} P \in End_{\mathbb{C}}(\mathcal{O}_X(U)) \mid \\ {[P, \mathcal{D}_X(U)(0)] \subset \mathcal{D}_X(U)(n-1)} \end{array} \right\}$$

  where $[\ ,\ ]$ *is the operator commutator.*

Note that $\mathcal{D}_X(U)$ is filtered by the subrings $\mathcal{D}_X(U)(n)$. The associated graded ring can be canonically identified with the commutative ring $\mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$.

We now turn to the global situation. On the complex manifold X we have the ring sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ of $\mathbb{C}$-linear endomorphisms of $\mathcal{O}_X$. Let $\mathcal{D}_X(0) = \mathcal{O}_X$ and define recursively

$$\mathcal{D}_X(n) := \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid [P, \mathcal{D}_X(U)(0)] \subset \mathcal{D}_X(n-1)\}$$

and $\mathcal{D}_X := \cup_{n=0}^{\infty} \mathcal{D}_X(n)$. Then $\mathcal{D}_X$ is called the sheaf of holomorphic differential operators on X. As before $\mathcal{D}_X$ is filtered by the subsheaves $\mathcal{D}_X(n)$ and it can be shown that the associated graded sheaf is isomorphic to $\mathcal{O}_X[\xi_1, \ldots, \xi_n]$. This can also be expressed as follows:

Lemma B.2. *The graded sheaf $gr(\mathcal{D}_X)$ associated to the above filtration is isomorphic to the subsheaf of $\pi_*(\mathcal{O}_{T^*X})$ which consists of functions which are polynomial in the fibers of $\pi : T^*X \to X$.*

We quote another fundamental result. The somewhat technical proof relies essentially on the corresponding result for the sheaf $\mathcal{O}_X$ (Oka's lemma).

Proposition B.3. *$\mathcal{D}_X$ is a coherent sheaf of rings, that is, for each open set $U \subset X$ and each morphism*

$$\varphi : \mathcal{D}^p_{X\,|U} \longrightarrow \mathcal{D}^q_{X\,|U}$$

the sheaf $\mathcal{K}er(\varphi)$ is locally of finite type.

As already said, differential systems on a manifold X can be represented as a module over $\mathcal{D}_X$. Here we explain this correspondence.

Consider any coherent module $\mathcal{M}$ over $\mathcal{D}_X$. Coherence implies that for each $U \subset X$ there is a presentation:

$$\mathcal{D}^p_{X\,|U} \xrightarrow{\varphi} \mathcal{D}^q_{X\,|U} \longrightarrow \mathcal{M} \longrightarrow 0$$

The morphism φ corresponds to a matrix $(A)_{i,j}$ with differential operators as entries. This means that the generators m_j of $\mathcal{M}$ satisfy:

$$\sum_{j=1}^{p} A_{i,j} m_j = 0 \quad \forall i = 1 \ldots q$$

Thus we see that solving the system of differential equations given by the matrix A is equivalent to associating a function (say holomorphic) to each m_j, so to giving a $\mathcal{D}_X$-linear homomorphism from $\mathcal{M}$ to $\mathcal{O}_X$ (remark that $\mathcal{O}_X$ is naturally a $\mathcal{D}_X$-module by ordinary differentiation). So a differential system corresponds to a $\mathcal{D}_X$-module $\mathcal{M}$ and its holomorphic solutions are given by the sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. The advantage of this description is that it is independent of any choice, whereas a differential system can have several representation (e.g., a single differential equation of degree n can always be transformed into a system of n first-order equations).

The next step consists in studying filtrations on $\mathcal{D}_X$-modules which are in some sense compatible with the natural filtration on $\mathcal{D}_X$. These are called "good" and defined as follows.

Definition B.4. *Let $\mathcal{M}$ be a given coherent $\mathcal{D}_X$-module. A good filtration of $\mathcal{M}$ is given by submodules $(\mathcal{M}_k)_{k\in\mathbb{N}}$ such that*

- $\mathcal{M}_k \subset \mathcal{M}_{k+1}$ *and* $\mathcal{D}_X(n)\mathcal{M}_k \subset \mathcal{M}_{k+n}$ *for all* $n, k \in \mathbb{N}$
- $\mathcal{M} = \bigcup_{k\in\mathbb{N}} \mathcal{M}_k$
- *each* $\mathcal{M}_k$ *is* $\mathcal{O}_X$*-coherent*
- *There is* $N \in \mathbb{N}$ *such that*

$$\mathcal{D}_X(n)\mathcal{M}_N = \mathcal{M}_{n+N}$$

  for all $n \in \mathbb{N}$

By the very definition of coherence, any such $\mathcal{D}_X$-module admits locally a good filtration (take the filtration induced by the standard filtration of $\mathcal{D}_X^k$ with $\mathcal{D}_X^k \twoheadrightarrow \mathcal{M}$). It is not clear under which circumstances a globally defined good filtration exist. However, it is known that for *holonomic* modules there is always a global good filtration.

Now we will define the geometric object which relates $\mathcal{D}$-modules to lagrangian subvarieties. Consider a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and a good filtration $(\mathcal{M}_k)$ over some open set U. Then $gr(M)_{|U}$ is a module over $gr(\mathcal{D}_X)_{|U}$, thus, we can define the *annihilator* of $gr(M)_{|U}$ in $gr(\mathcal{D}_X)_{|U}$, which is a coherent sheaf of ideals of $gr(\mathcal{D}_X)_{|U}$. Now the crucial fact is that although this annihilator ideal depends on the chosen locally good filtration, its radical is an invariant of $\mathcal{M}_{|U}$ which can therefore be glued into an ideal of $gr(\mathcal{D}_X)$. More precisely, the following holds.

Theorem B.5. *There is a sheaf of ideals in $gr(\mathcal{D}_X)$, which is denoted by $\sqrt{gr(\mathcal{M})}$ such that on each restriction to an open subset U where $\mathcal{M}_{|U}$ has a good filtration we have*

$$\sqrt{gr(\mathcal{M})}_{|U} = rad\left(ann_{gr(\mathcal{D}_X)_{|U}}\left(gr(\mathcal{M})_{|U}\right)\right)$$

As we said in lemma B.2 on page 175, $gr(\mathcal{D}_X)$ is closely related to $\mathcal{O}_{T^*X}$. In particular, $\mathcal{O}_{T^*X}$ is a flat module over $\pi^{-1}(gr(\mathcal{D}_X))$ (this is easily to be seen true at every point of X). Thus we have the inclusion

$$\pi^{-1}\left(\sqrt{(gr(\mathcal{M}))}\right) \otimes_{\pi^{-1}(gr(\mathcal{D}_X))} \mathcal{O}_{T^*X} \hookrightarrow \mathcal{O}_{T^*X}$$

The ideal in $\mathcal{O}_{T^*X}$ generated in this way defines an analytic subset of the holomorphic cotangent bundle. This is the *characteristic variety* attached to the coherent $\mathcal{D}_X$-module $\mathcal{M}$. Usual notations for this space are $char(\mathcal{M})$ or $SS(\mathcal{M})$ (the latter symbol refers to the name "singular support", which is justified from the microlocal viewpoint).

Proposition B.6. *The characteristic variety $char(\mathcal{M})$ is a coisotropic subset of the symplectic manifold T^*X, i.e., the Poisson bracket of two elements of the defining ideal*

$$\pi^{-1}\left(\sqrt{(gr(\mathcal{M}))}\right) \otimes_{\pi^{-1}(gr(\mathcal{D}_X))} \mathcal{O}_{T^*X}$$

lies still in that ideal.

There are at least two different proofs of this result. One uses microlocal techniques, the other one, due to Gabber, is a far more general result on filtered rings and modules over them (see [Gab81] and [Bjö93]). We remark that Gabber's proof can be generalized in the context of differential operators constructed from Lie algebroids, see section 3.1.1.

B.2 Holonomic $\mathcal{D}_X$-modules

As we said in the last section, a characteristic variety is always coisotropic. This implies that $dim\,(char(\mathcal{M})) \geq n$ where n is the dimension of the underlying variety X. Note that this *Bernstein inequality* is proved independently of the involutiveness of $char(\mathcal{M})$.

Definition B.7. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. $\mathcal{M}$ is called holonomic iff its characteristic variety is of dimension n, i.e., if it is a lagrangian subvariety of T^*X.*

According to this definition, holonomic $\mathcal{D}_X$-modules provide examples for lagrangian subvarieties. The simplest lagrangian submanifold of the cotangent bundle is its zero section. It is easy to show that iff the characteristic variety is just the zero section, then the good filtration is stationary which in turn implies that the $\mathcal{D}_X$-module is $\mathcal{O}_X$-coherent. Then it is even locally free over $\mathcal{O}_X$ and its $\mathcal{D}_X$-module structure is nothing else than an integrable connection.

In general, the characteristic variety is much more complicated. But at least we have the following relation with the conormal space construction.

Lemma B.8. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let $\pi : T^*X \to X$ be the projection. Denote by $C_{\mathcal{M}}$ the union of the components of char$(\mathcal{M})$ which are different from the zero-section of π. Then we have*

$$C_{\mathcal{M}} = \bigcup_{Z \subset \pi(C_{\mathcal{M}})} T^*_Z X$$

where Z runs over the irreducible components of $\pi(C_{\mathcal{M}})$.

Its well known that flat connections on vector bundles (i.e., locally free $\mathcal{O}_X$-modules) are in one to one bijection with local systems on X (which in turn are equivalent to representations of the fundamental group). The so called *Riemann-Hilbert-correspondence* determines the class of holonomic $\mathcal{D}_X$-modules to which this fact can be generalized. The first essential step is Kashiwara's constructibility theorem. We include this fundamental result here in order to motivate one of our central theorems on deformations of lagrangian singularities (see 3.35 on page 88). We will use some notions from complex analysis concerning stratifications. See for example [Mer93].

Theorem B.9. *Let $\mathcal{M}$ be $\mathcal{D}_X$ coherent and holonomic. Then there is a Whitney regular stratification of X such that the* solution complex *of $\mathcal{M}$*

$$\mathcal{S}ol^\bullet(\mathcal{M}) := \mathcal{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

is constructible with respect to this stratification.

Remarks:

- Constructibility of a sheaf means that the restriction of this sheaf to each stratum is a local system of finite dimensional vector spaces over $\mathbb{C}$.
- The solution complex $\mathcal{S}ol^\bullet(\mathcal{M})$ is seen as an object in the derived category of sheaves of complex vector spaces on X. Therefore, constructibility of such a complex means constructibility of its cohomology sheaves.
- We could have considered the sheaf complex

$$DR^\bullet(\mathcal{M}) := \mathcal{RH}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$$

  instead, which is called the *de Rham complex* of $\mathcal{M}$. Then constructibility holds as well. But this can be deduced more generally from the duality theorem for holonomic modules.
- In the definition of the solution complex of $\mathcal{M}$ (as well as in that of the de Rham complex) we do not actually use the fact that $\mathcal{M}$ is a single holonomic module, that is, we can state the same theorem for complexes of holonomic modules (i.e. complexes of $\mathcal{D}_X$-modules such that their cohomologies are holonomic). It follows from general consideration about derived categories that the proof of constructibility in this case is almost the same as for single degree complexes.

We will only give an idea of the proof following [Bjö93] and skip the technical details. We will use (but not prove) the fact that the spaces $Z \subset \pi(char(\mathcal{M}))$ provides a Whitney stratification of X. Then first we show that the restrictions

$$\mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Z}$$

for each $p \in \mathbb{N}$ and $Z \subset \pi(char(X))$ form a local system. The second step consists in proving that the stalk of $\mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ at each point is finite-dimensional. The essential ingredient for both steps is the following result from functional analysis whose proof can be found in [KV71].

Proposition B.10. *Consider two (bounded) complexes of Fréchet spaces with continuous linear differentials. Suppose that we are given a morphism of these complexes consisting of compact operators. If, under these hypotheses, the mapping is a quasi-isomorphism, then the cohomology of the two complexes are finite-dimensional vector spaces*

The second technical result (which is needed to use the preceding construction) concerns the restriction morphism of a holonomic $\mathcal{D}_X$-module with respect to C^1-domains with *non-characteristic boundary.*

Definition B.11. *Let φ a real valued function of class C^1 and consider the domain $\Omega = \{x \in X \mid \varphi(x) < 0\}$. Suppose furthermore that $\partial\Omega$ is compact and that φ is regular there. Then we set*

$$N^*_\Omega = \{(x, \partial\varphi(x)) \mid x \in \partial\Omega\}$$

where $\partial\varphi$ is the holomorphic differential of φ. We say that Ω is non-characteristic with respect to some holonomic $\mathcal{D}_X$-module $\mathcal{M}$ iff

$$char(\mathcal{M}) \cap N^*_\Omega = \emptyset$$

We note the following important fact which is used two times in the proof of the constructibility theorem: Consider a function φ as above. Then for any regular real subspace Z the set of values c of φ such that $x \in \varphi^{-1}(c) \cap Z$ and $(x, \partial\varphi(x)) \in T^*_Z X$ is finite.

The technical result which is needed for the proof is as follows.

Lemma B.12. *Let a family of C^1-domains Ω_t with $t \in [0,1]$ be given such that*

$$\Omega_t = \bigcup_{s<t} \Omega_s \quad \textit{and} \quad \overline{\Omega}_t = \bigcap_{s>t} \Omega_s$$

and suppose that all Ω_t are non-characteristic with respect to $\mathcal{M}$. Then each of the restriction morphisms

$$\mathbb{H}^p(\Omega_1, Sol^\bullet(\mathcal{M})) \longrightarrow \mathbb{H}^p(\Omega_t, Sol^\bullet(\mathcal{M}))$$

is an isomorphism.

A proof can be found in [Bjö93]. Note however that it uses microlocal techniques in order to obtain a vanishing result for certain local cohomology groups.

Proof of the theorem. As the result is local in nature, we can assume that X is an embedded in some $\mathbb{C}^n$. Let $Z \subset \pi(char(M))$ be a component and $x_0 \in Z$ a point. We consider the restriction

$$\mathcal{F} := \mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Z \cap B_{x_0}(\epsilon)}$$

where $B_{x_0}(\epsilon)$ is a small ϵ-ball around x_0 inside X. We have to show that $\mathcal{F}$ is a constant sheaf. Define for any $x \in Z \cap B_{x_0}(\epsilon)$ and any $t \in (0,1)$ the set $\Omega_t(x) := \{y \in B_{x_0}(\epsilon) \ : \ |(1-t)x - ty - x_0| < \epsilon t\}$. We have $\Omega_1(x) = B_{x_0}(\epsilon)$ for any x. Moreover, it can be shown that there is an ϵ_0 such that $N^*_{\partial\Omega_t(x)}$ does not meet the conormal cone to $\pi(char(\mathcal{M}))$ for any t and $x \in Z \cap B_{x_0}(\epsilon_0)$. This implies that $\partial\Omega_t(x)$ is non-characteristic with respect to $\mathcal{M}$, which makes it possible to apply lemma B.12 on the preceding page to get that the restrictions

$$\mathbb{H}^p(B_{x_0}(\epsilon_0), Sol^\bullet(\mathcal{M})) \longrightarrow \mathbb{H}^p(\Omega_t(x), Sol^\bullet(\mathcal{M}))$$

are isomorphisms. By letting $t \to 0$, we obtain that the stalk $\mathcal{F}_x$ is equal to $\mathbb{H}^p(B_{x_0}(\epsilon_0), Sol^\bullet(\mathcal{M}))$. Therefore, $\mathcal{F}$ is constant.

For the second part, i.e., the finiteness of the stalks $\mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{x_0}$, we use a similar argument: There is an ϵ such that $\partial B_{x_0}(\epsilon')$ is non-characteristic with respect to $\mathcal{M}$ for every $\epsilon' < \epsilon$. Now we consider the family $\Omega_t(x_0) := B_{x_0}(t\epsilon)$. Then the desired result follows immediately from lemma B.12 and proposition B.10 on the facing page. □

Now that we have seen that $\mathcal{S}ol$ defines in fact a functor from the category of holonomic $\mathcal{D}$-modules to constructible sheaves one might ask whether this functor is an equivalence. It turns out that this is the case when we restrict this functor to a subclass consisting of *regular* holonomic modules. Recall first the case where X is one-dimensional. Then the singular locus of $\mathcal{M}$, i.e., the components of $\pi(char(X))$ which are of dimension less then n, is a (possibly empty) discrete set of points. Outside of these points $\mathcal{M}$ is a connection in the sense described above. Then the *localized* module $\mathcal{M}[t^{-1}]$ (t being a coordinate around the singular point) is called a meromorphic connection with a regular singularity if there is a basis of this module over $\mathbb{C}\{t\}[t^{-1}]$ such that the matrix of the connection with respect to this basis has a pole of order at most one. It is known that regular singular connections are in one-to-one correspondence to local systems on the punctured disc. Consequently, one possible definition of a regular holonomic module $\mathcal{M}$ is that the pull-back (which is defined in the category of $\mathcal{D}_X$-modules) to any curve is regular in the sense just described. The next definition makes this precise and presents equivalent definitions of regularity.

Theorem B.13. *The following conditions are equivalent.*

1. *Let $\gamma : C \to X$ holomorphic, where C is smooth and one-dimensional. Then the complex $\gamma^+(\mathcal{M})$, where γ^+ is the pull-back functor in the category of coherent $\mathcal{D}_X$-modules is regular, i.e., its localization around each singular point is a meromorphic connection.*

2. *There is a globally defined good filtration on $\mathcal{M}$ such that the annihilator of $gr(\mathcal{M})$ in $gr(\mathcal{D}_X)$ is a radical ideal (Note that for each holonomic $\mathcal{D}_X$-module there exists a globally defined good filtration).*

3. *Denote by $\widehat{\mathcal{O}}_{X,x}$ the completion of the local ring at a point $x \in X$. Then we have for all points x:*

$$RHom_{\mathcal{D}_{X,x}}(\mathcal{M}_x, \widehat{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}) = 0$$

   that is, the formal and the analytic solution complex coincides.

A holonomic module satisfying one of these condition is called regular holonomic.

With this definition in mind we can state the Riemann-Hilbert correspondence.

Theorem B.14. *The functor DR is an equivalence from the derived category of complexes of regular holonomic $\mathcal{D}_X$-modules to the derived category of complexes of constructible sheaves of $\mathbb{C}$-vector spaces.*

Remark: In the third chapter, we study a sheaf complex arising from a lagrangian singularity. It turns out that the cohomology of this complex is constructible under some hypothesis. Therefore, by the Riemann-Hilbert correspondence, there is a (complex of) holonomic $\mathcal{D}$-module(s) corresponding to it via the functor DR.

Bibliography

[AGZV85] V.I. Arnol'd, S.M. Gusein-Zade, and A.N. Varchenko, *Singularities of differentiable maps. Volume I: The classification of critical points, caustics and wave fronts.*, Monographs in Mathematics, vol. 82, Birkhäuser, 1985.

[AGZV88] ______, *Singularities of differentiable maps. Volume II: Monodromy and asymptotics of integrals.*, Monographs in Mathematics, vol. 83, Birkhäuser, 1988.

[Arn82] V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays, and the open swallowtail.*, Funct. Anal. Appl. **15** (1982), 235–246.

[Arn90] ______, *Singularities of caustics and wave fronts*, Mathematics and Its Applications (Soviet Series), vol. 62, Kluwer Academic Publishers, 1990.

[Art76] Michael Artin, *Lectures on deformations of singularities. Notes by C. S. Seshadri, Allen Tannenbaum*, Lectures on Mathematics and Physics, vol. 54, Tata Institute of Fundamental Research, Bombay, 1976.

[Aud98a] Michèle Audin, *An introduction to Frobenius manifolds, moduli spaces of stable maps and quantum cohomology*, Preprint IRMA Strasbourg, 1998.

[Aud98b] ______, *Symplectic geometry in Frobenius manifolds and quantum cohomology.*, J. Geom. Phys. **25** (1998), no. 1-2, 183–204.

[Ban94] A. Banyaga, *An introduction to symplectic geometry*, Holomorphic curves in symplectic geometry (Michèle Audin and Jacques Lafontaine, eds.), Progress in Mathematics, vol. 117, Birkhäuser, 1994, pp. 17–40.

[BB93] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand seminar. Part 2: Papers of the Gelfand seminar in functional analysis held at Moscow University, Russia, September 1993. (Sergej Gelfand and Simon Gindikin, eds.), Advances in Soviet Mathematics, vol. XIV, American Mathematical Society, 1993, p. 208.

[BF96] Ragnar-Olaf Buchweitz and Hubert Flenner, *Unfoldings of holomorphic maps as deformations.*, Complex analysis and geometry. Proceedings of the conference held in Trento, Italy, June 5-9, 1995. (Vincenzo Ancona, Edoardo Ballico, and Alessandro Silva, eds.), Lecture Notes in Pure and Applied Mathematics, vol. 173, Marcel Dekker, 1996, p. 554.

[BF00] ______, *Deformation Theory*, preprint, 2000.

[BG80] Ragnar-Olaf Buchweitz and Gert-Martin Greuel, *The Milnor number and deformations of complex curve singularities*, Invent. Math. **58** (1980), 241–281.

[Bjö93] Jan-Erik Björk, *Analytic $\mathcal{D}$-modules and applications*, Mathematics and its Applications, vol. 247, Kluwer Academic Publishers, 1993.

[BK98] Sergey Barannikov and Maxim Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields.*, Int. Math. Res. Not. **1998** (1998), no. 4, 201–215.

[BM97] Ragnar-Olaf Buchweitz and John J. Millson, *CR-geometry and deformations of isolated singularities.*, Mem. Am. Math. Soc. **597** (1997), 96.

[Buc81] Ragnar-Olaf Buchweitz, *Contributions à la théorie des singularités: 1) Déformations de diagrammes, déploiements et singularités très rigides, liaison algébrique; 2) Sur le critère d'équisingularité de Zariski et courbes monomiales non lissifiables*, Thèse d'Etat, Mathématiques, Université Paris 7, 1981.

[CdV01] Yves Colin de Verdière, *Singular langrangian manifolds and semi-classical analysis*, preprint Institut Fourier, Grenoble, 2001.

[CJNMM96] Francisco J. Castro-Jiménez, Luis Narváez-Macarro, and David Mond, *Cohomology of the complement of a free divisor*, Trans. Am. Math. Soc. **348** (1996), no. 8, 3037–3049.

[dJP00] Theo de Jong and Gerhard Pfister, *Local analytic geometry. Basic theory and applications.*, Advanced Lectures in Mathematics, Vieweg, 2000.

[Dub96] Boris Dubrovin, *Geometry of 2D topological field theories.*, Integrable systems and quantum groups. Lectures given at the 1st session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 14-22, 1993. (M. Francaviglia and S. Greco, eds.), Lecture Notes in Mathematics, vol. 1620, Springer-Verlag, 1996, p. 488.

[EGSS02] David Eisenbud, Daniel R. Grayson, Michael Stillman, and Bernd Sturmfels (eds.), *Computations in algebraic geometry with Macaulay 2.*, Algorithms and Computation in Mathematics, vol. 8, Springer, 2002.

[Eis95] David Eisenbud, *Commutative algebra. With a view toward algebraic geometry.*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, 1995.

[FM98] Barbara Fantechi and Marco Manetti, *Obstruction calculus for functors on Artin Rings, I*, Journal of Algebra **202** (1998), 541–576.

[FM99] ______, *On the T^1-lifting theorem*, Journal of Algebraic Geometry **8** (1999), 31–39.

[Gab81] Ofer Gabber, *The integrability of the characteristic variety.*, Am. J. Math. **103** (1981), 445–468.

[Gar02] Mauricio Garay, *Stable complete integrability and singular lagrangian varieties*, preprint Johannes-Gutenberg-Universität Mainz, Fachbereich Mathematik, 2002.

[GG80] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities. 2nd corr. printing.*, Graduate Texts in Mathematics, vol. 14, Springer-Verlag, 1980.

[Giv86] A.B. Givental', *Lagrangian imbeddings of surfaces and unfolded Whitney umbrella*, Funct. Anal. Appl. **20** (1986), 197–203.

[Giv88] ______, *Singular Lagrangian manifolds and their Lagrangian mappings*, Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. **33** (1988), 55–112, English translation: J. Soviet Math. 52.4 (1990), 3246-3278.

[GM93] Michel Granger and Philippe Maisonobe, *A basic course on differential modules*, in Maisonobe and Sabbah [MS93], pp. 103–168.

[GP02] Gert-Martin Greuel and Gerhard Pfister, *A singular introduction to commutative algebra. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann.*, Springer, 2002.

[GPS01] G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR *2.0*, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001, `http://www.singular.uni-kl.de`.

[Gre80] Gert-Martin Greuel, *Dualität in der lokalen Kohomologie isolierter Singularitäten.*, Math. Ann. **250** (1980), 157–173.

[Gro97] Mark Gross, *Deforming Calabi-Yau threefolds.*, Math. Ann. **308** (1997), no. 2, 187–220.

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2.

[GS88] Murray Gerstenhaber and Samuel D. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications, Based on the NATO Advanced Study Institute held at “Il Ciocco”, Castelvecchio-Pascoli, Tuscany, June 1-14, 1986 (Michiel Hazewinkel and Murray Gerstenhaber, eds.), NATO ASI Series, Series C: Mathematical and Physical Sciences, vol. 247, Kluwer Academic Publishers Group, 1988, pp. 11–264.

[GvS02] Mauricio Garay and Duco van Straten, *On the topology of lagrangian Milnor fibers*, preprint Johannes-Gutenberg-Universität Mainz, Fachbereich Mathematik, 2002.

[Her02] Claus Hertling, *Frobenius Manifolds and Moduli Spaces for Singularities*, Cambridge Tracts in Mathematics, vol. 151, Cambridge University Press, 2002.

[HL82] Reese Harvey and H. Blaine Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[Ill71] Luc Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, vol. 239, Springer-Verlag, 1971.

[Ill72] ———, *Complexe cotangent et déformations. II*, Lecture Notes in Mathematics, vol. 283, Springer-Verlag, 1972.

[Ish92] Goo Ishikawa, *The local model of an isotropic map-germ arising from one-dimensional symplectic reduction*, Math. Proc. Camb. Philos. Soc. **111** (1992), no. 1, 103–112.

[Ish96] Go-o Ishikawa, *Symplectic and Lagrange stabilities of open Whitney umbrellas*, Invent. Math. **126** (1996), no. 2, 215–234.

[Joy00] D. Joyce, *Singularities of special Lagrangian fibrations and the SYZ Conjecture*, preprint dg/0011179 (2000), 54.

[Käl98] Rolf Källström, *Smooth modules over Lie algebroids I*, preprint alg-geom/9808108, 1998.

[KV71] Reinhardt Kiehl and Jean-Louis Verdier, *Ein einfacher Beweis des Kohärenzsatzes von Grauert*, Math. Ann. **195** (1971), 24–50.

[Loo84] E.J.N. Looijenga, *Isolated singular points on complete intersections.*, Lecture Note Series, vol. 77, Cambridge University Press, 1984.

[Mal74] Bernard Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. Ec. Norm. Super., IV. Ser. **7** (1974), 405–430.

[Man98] Marco Manetti, *Deformation Theory via Differential Graded Lie Algebras*, electronic version available via http://venere.mat.uniroma1.it/people/manetti/, 1998.

[Man99] ———, *Extended Deformation Functors*, preprint alg-geom/9910071, 1999.

[Man01a] ———, *Deformations of singularities via differential graded Lie algebras*, electronic version available via http://venere.mat.uniroma1.it/people/manetti/, 2001.

[Man01b] ———, *Lectures on deformations of complex manifolds*, electronic version available via http://venere.mat.uniroma1.it/people/manetti/, 2001.

[Meb89] Zoghman Mebkhout, *Systèmes différentiels. Le formalisme des six opérations de Grothendieck pour les $\mathcal{D}_X$-modules cohérents*, Travaux en Cours, vol. 35, Hermann, 1989.

[Mer93] Michel Merle, *On some points of local analytic geometry*, in Maisonobe and Sabbah [MS93], pp. 81–101.

[Mor65] Bernard Morin, *Formes canoniques des singularités d'une application différentiable*, C. R. Acad. Sci. **260** (1965), no. 2, 5662–5665, 6503–6506.

[MS93] Philippe Maisonobe and Claude Sabbah (eds.), *$\mathcal{D}$-modules cohérent et holonomes*, Travaux en cours, vol. 46, Hermann, 1993.

[Nam94] Yoshinori Namikawa, *On deformations of Calabi-Yau 3-folds with terminal singularities.*, Topology **33** (1994), no. 3, 429–446.

[Oda87] Tadao Oda, *K. Saito's period map for holomorphic functions with isolated critical points*, Algebraic geometry, Sendai, 1985. (Papers presented at the Symposium on Algebraic Geometry held at Tohoku University, Sendai, Japan, June 24-29, 1985). (Tadao Oda, ed.), Advanced Studies in Pure Mathematics, vol. 10, North-Holland, 1987, pp. 591–648.

[Pha79] Frederic Pham, *Singularités des systèmes differentiels de Gauss-Manin*, Progress in Mathematics, vol. 2, Birkhäuser, 1979.

[Ran92] Ziv Ran, *Deformations of manifolds with torsion or negative canonical bundle.*, J. Algebr. Geom. **1** (1992), no. 2, 279–291.

[Rin63] G.S. Rinehart, *Differential forms on general commutative algebras*, Trans. Am. Math. Soc. **108** (1963), 195–222.

[Sab02] Claude Sabbah, *Déformations isomonodromiques et variétés de Frobenius*, Savoirs Actuels, vol. 16, EDP Sciences, 2002.

[Sai71] Kyoji Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math. **14** (1971), 123–142.

[Sch68] M. Schlessinger, *Functors of Artin rings*, Trans. Am. Math. Soc. **130** (1968), 208–222.

[Sev99] Christian Sevenheck, *Lagrange-Singularitäten und ihre Deformationen*, Diplomarbeit, Heinrich-Heine-Universität Düsseldorf, 1999.

[Ste03] Jan Stevens, *Deformations of Singularities*, Lecture Notes in Mathematics, vol. 1811, Springer, 2003.

[Tei77] Bernard Teissier, *The hunting of invariants in the geometry of discriminants. Five lectures at the Nordic Summer School*, Real and complex singularities, Oslo 1976. Proceedings of the Nordic Summer School/NAVF. Symposium in Mathematics Oslo, August 5-25, 1976. (Per Holm, ed.), Sijthoff & Noordhoff International Publishers, 1977, pp. 565–678.

[Var83] A.N. Varchenko, *On semicontinuity of the spectrum and an upper estimate for the number of singular points of a projective hypersurface.*, Sov. Math., Dokl. **27** (1983), 735–739.

[VG82] A.N. Varchenko and A.B. Givental', *Mapping of periods and intersection form.*, Funct. Anal. Appl. **16** (1982), 83–93.

[vS87] D. van Straten, *On the Betti numbers of the Milnor fibre of a certain class of hypersurface singularities*, Singularities, representation of algebras, and vector bundles (Gert-Martin Greuel and Günter Trautmann, eds.), Lect. Notes Math., vol. 1273, Springer-Verlag, 1987, pp. 203–220.

[Wei73] Alan Weinstein, *Lagrangian submanifolds and Hamiltonian systems.*, Ann. of Math., II. Ser. **98** (1973), 377–410.

[Zak90] V.M. Zakalyukin, *Generating ideals of Lagrangian varieties*, Theory of singularities and its applications (V.I. Arnol'd, ed.), Advances in Soviet Mathematics, vol. 1, American Mathematical Society, 1990, p. 333.